Chapter 3

Statics

3.1 Basic Concepts

The study of statics is based on six fundamental principles. These are

- Newton's three fundamental laws,
- parallelogram law for the addition of forces,
- principle of transmissibility,
- Newton's law of gravitation.

For details see [1], [2], [3].

3.1.1 Force

A force represents the action of one body on another. This action can be realized by actual contact or by action at a distance (e.g. gravitational force). A force is represented by a *vector*. It is characterized by its magnitude, its point of action, and its direction. We should distinguish three kinds of vectors, namely *a free vector*, *a fixed vector*, and a vector *bound to its line of action*. All of these vectors have their place in mechanics. We will deal with rigid body mechanics in this chapter.

The addition of two forces acting at the same point of action is governed by the *parallelogram law*. This states that two forces may be replaced by a single force (called the *resultant*) obtained as the diagonal of the parallelogram the sides of which are the given forces (see Fig. 3.1).

The following is valid

$$F_{\mathrm{r}} = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos\alpha_2} \tag{3.1}$$

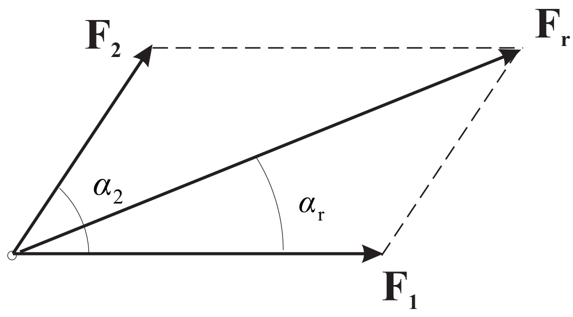

Figure 3.1: Parallelogram law.

$$\tan\alpha_{\mathrm{r}} = \frac{F_2 \sin\alpha_2}{F_1 + F_2\cos\alpha_2} \tag{3.2}$$

Consider a force **F** acting at the origin of the Cartesian (rectangular and right-handed) coordinate system (see Fig. 3.2).

The force **F** may be resolved into three *components*

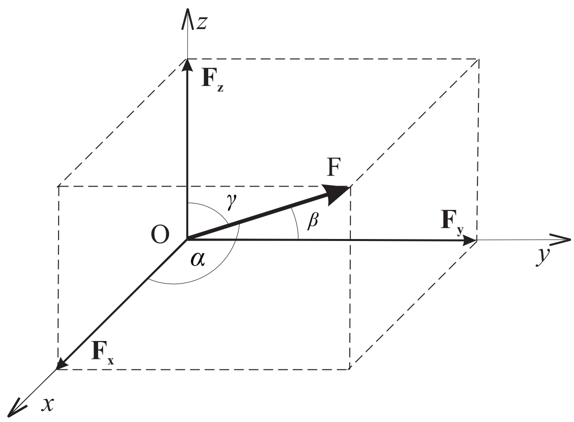

Figure 3.2: Decomposition of a force into three components.

$$F_x = F\cos\alpha, \quad F_y = F\cos\beta, \quad F_z = F\cos\gamma \tag{3.3}$$

$$F = |\mathbf{F}| = \sqrt{F_x^2 + F_y^2 + F_z^2} \tag{3.4}$$

Introducing the unit vectors $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ directed respectively along x, y, and z axes, we may express **F** in the form

$$\mathbf{F} = F_x\,\mathbf{i} + F_y\,\mathbf{j} + F_z\,\mathbf{k} \tag{3.5}$$

or

$$\mathbf{F} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = [F_x, F_y, F_z]^{\mathrm{T}} \tag{3.6}$$

If the components F_x, F_y, F_z of a force $\mathbf{F}$ are given then *the magnitude* F of the force is obtained from (3.4) and *the direction cosines* are

$$\cos\alpha = \frac{F_x}{F}, \qquad \cos\beta = \frac{F_y}{F}, \qquad \cos\gamma = \frac{F_z}{F} \tag{3.7}$$

Given n *concurrent forces*, we may determine the resultant $\mathbf{F}_{\mathrm{r}}$ by summing their rectangular components:

$$F_{\mathrm{r}x} = \sum_{i=1}^{n} F_{ix}, \quad F_{\mathrm{r}y} = \sum_{i=1}^{n} F_{iy}, \quad F_{\mathrm{r}z} = \sum_{i=1}^{n} F_{iz} \tag{3.8}$$

$$\mathbf{F}_{\mathrm{r}} = F_{\mathrm{r}x}\,\mathbf{i} + F_{\mathrm{r}y}\,\mathbf{j} + F_{\mathrm{r}z}\,\mathbf{k} \tag{3.9}$$

$$\cos\alpha_{\mathrm{r}} = \frac{F_{\mathrm{r}x}}{F_{\mathrm{r}}}, \qquad \cos\beta_{\mathrm{r}} = \frac{F_{\mathrm{r}y}}{F_{\mathrm{r}}}, \qquad \cos\gamma_{\mathrm{r}} = \frac{F_{\mathrm{r}z}}{F_{\mathrm{r}}} \tag{3.10}$$

3.1.2 Moment of a force about a point

The *moment of a force about a point* O is defined as the *vector product* (cross product)

$$\mathbf{M}_{\mathrm{O}} = \mathbf{r} \times \mathbf{F} \tag{3.11}$$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is *the radius vector* (position vector) drawn from O to the point of application A of the force $\mathbf{F}$, and $\mathbf{F} = F_x\,\mathbf{i} + F_y\,\mathbf{j} + F_z\,\mathbf{k}$ is the force vector acting on A. Both vectors are expressed by their components (x, y, z) and (F_x, F_y, F_z), respectively, in Cartesian coordinate system.

Denoting by φ the angle between $\mathbf{r}$ and $\mathbf{F}$, we find that the magnitude of the moment of $\mathbf{F}$ about O may be expressed as

$$M_{\mathrm{O}} = r\,F\sin\varphi = F\,d \tag{3.12}$$

where d is the perpendicular distance from O to the line of action of $\mathbf{F}$.

In matrix calculus (and in Matlab as well) we use the expression

$$\mathbf{M}_{\mathrm{O}} = \hat{\mathbf{r}}\,\mathbf{F} \tag{3.13}$$

where

$$\hat{\mathbf{r}} = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \tag{3.14}$$

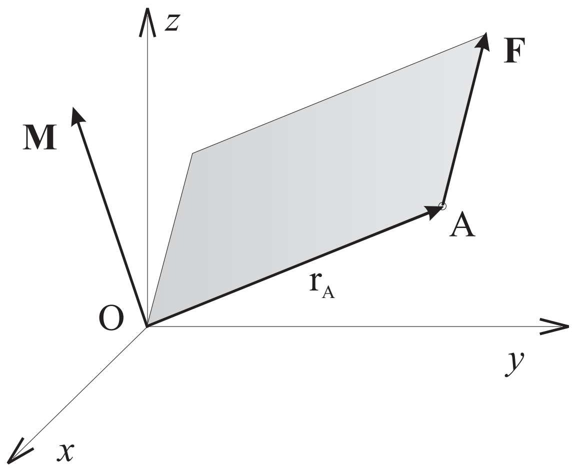

Figure 3.3: Moment of the force **F** about the point O.

is the *skew-symmetric matrix* representation of the vector **r** and

$$\mathbf{F} = [F_x, F_y, F_z]^{\mathrm{T}} \tag{3.15}$$

is $(3, 1)$ - matrix representation of the vector **F**.

The Cartesian components of the moment $\mathbf{M}_{\mathrm{O}}$ of a force **F** can be found to be

$$\mathbf{M}_{\mathrm{O}} = \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} yF_z - zF_y \\ zF_x - xF_z \\ xF_y - yF_x \end{bmatrix} \tag{3.16}$$

The magnitude of $\mathbf{M}_{\mathrm{O}}$ is

$$M_{\mathrm{O}} = \sqrt{M_x^2 + M_y^2 + M_z^2} \tag{3.17}$$

The line of action of the moment $\mathbf{M}_{\mathrm{O}}$ is determined by direction cosines

$$\cos\alpha_{\mathrm{M}} = \frac{M_x}{M_{\mathrm{O}}}, \quad \cos\beta_{\mathrm{M}} = \frac{M_y}{M_{\mathrm{O}}}, \quad \cos\gamma_{\mathrm{M}} = \frac{M_z}{M_{\mathrm{O}}} \tag{3.18}$$

where $\alpha_{\mathrm{M}}, \beta_{\mathrm{M}}, \gamma_{\mathrm{M}}$ are angles between the line of action and coordinate axes x, y, z, respectively.

In a more general case of the moment about an arbitrary point B of a force **F** applied at A, we have

$$\mathbf{M}_{\mathrm{B}} = \hat{\mathbf{r}}_{\mathrm{AB}}\,\mathbf{F} \tag{3.19}$$

where $\hat{\mathbf{r}}_{\mathrm{AB}}$ is skew-symmetric matrix representation of the vector

$$\mathbf{r}_{\mathrm{AB}} = \mathbf{r}_{\mathrm{A}} - \mathbf{r}_{\mathrm{B}} \tag{3.20}$$

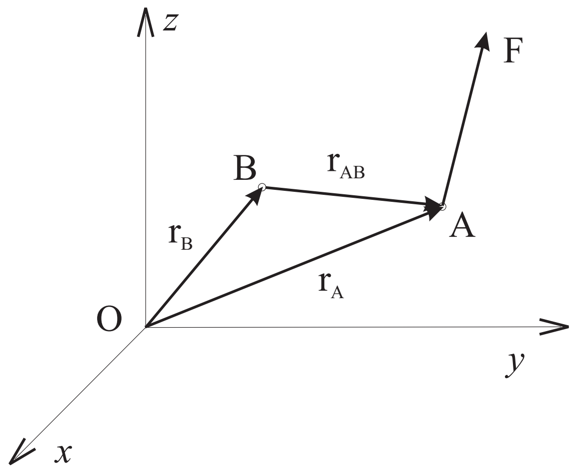

Figure 3.4: Moment of the force **F** about the point B.

Varignon's theorem

The moment about a given point O of the resultant of several concurrent forces is equal to the sum of the moments of the particular forces about the same point. This is

$$\mathbf{r} \times \mathbf{F}_1 + \mathbf{r} \times \mathbf{F}_2 + \cdots = \mathbf{r} \times (\mathbf{F}_1 + \mathbf{F}_2 + \ldots) \tag{3.21}$$

The theorem follows from distributive property of vector product.

3.2 Moment of a force about an axis

The moment of a force **F** *about an axis* p is defined as the projection OB on p of the moment $\mathbf{M}_\mathrm{O}$ (see Fig. 3.5).

Denoting $\boldsymbol{\lambda}$ the unit vector of p, the moment M_p of force **F** about an axis p can be expressed as *scalar product* (dot product)

$$M_\mathrm{p} = \boldsymbol{\lambda} \, . \, \mathbf{M}_\mathrm{O} \tag{3.22}$$

or as *the mixed triple product* of the unit vector $\boldsymbol{\lambda}$, the position vector $\mathbf{r}$, and the force **F**:

$$M_\mathrm{p} = \boldsymbol{\lambda} \, . \, (\mathbf{r} \times \mathbf{F}) \tag{3.23}$$

Using the determinant form for the mixed triple product, we have the magnitude of the moment

$$M_\mathrm{p} = \begin{vmatrix} \lambda_x & \lambda_y & \lambda_z \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix} \tag{3.24}$$

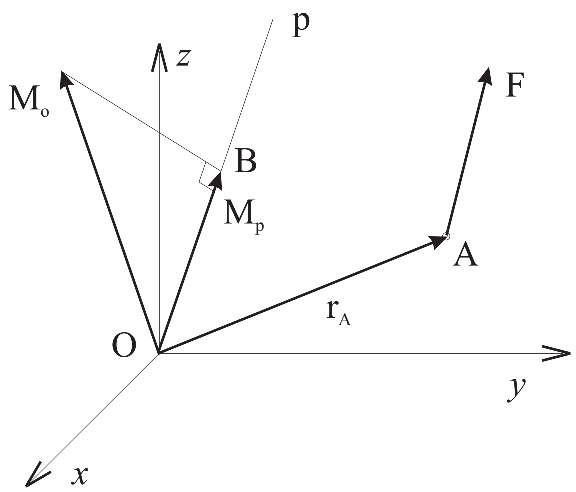

Figure 3.5: Moment of the force **F** about the axis p.

where	$\lambda_x, \lambda_y, \lambda_z$	are direction cosines of axis p
	x, y, z	are components of **r**
	F_x, F_y, F_z	are components of **F**

Exercise 3.2.1 Sample problem A force $F = 40$ N is applied at a point $\mathrm{M}(x_\mathrm{M}; y_\mathrm{M}; z_\mathrm{M}) \equiv \mathrm{M}(3; 2; 4)$ [m] (see Fig. 3.6). The line o_F of action of the force **F** is described by direction angles $(\alpha_F; \beta_F; \gamma_F) \equiv (80°; 60°;$ acute angle). An axis a which passes through the origin O has direction angles $(\alpha_\mathrm{a}; \beta_\mathrm{a}; \gamma_\mathrm{a}) \equiv (60°; 100°;$ acute angle). Determine:

- The moment $\mathbf{M}_\mathrm{O}$ of the force **F** about O.
- The moments M_x, M_y, and M_z of the force **F** about axes x, y, and z, respectively.
- The moment $\mathbf{M}_\mathrm{a}$ of the force **F** about a axis.

Notice: An acute angle α is such an angle for which the condition $0 < \alpha < 90°$ is valid.

Solution

We determine the angle γ_F first. Since the expression

$$\cos\alpha_F^2 + \cos\beta_F^2 + \cos\gamma_F^2 = 1$$

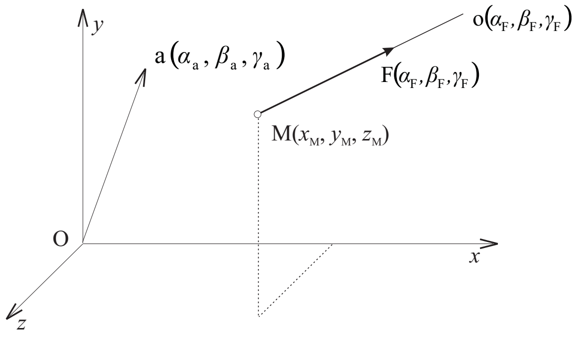

Figure 3.6: Exercise 3.2.1. Moment of the force **F** about the axis a.

is valid for any set of direction cosines, we have

$$\cos\gamma_F = +\sqrt{1-\cos^2\alpha_F-\cos^2\beta_F} = \sqrt{1-0.1736^2-0.5^2} = 0.848$$

According to the definition of the moment $\mathbf{M}_\mathrm{O}$ of the force **F**, we may write

$$\mathbf{M}_\mathrm{O} = \mathbf{r}_M \times \mathbf{F} = \hat{\mathbf{r}}_M\ \mathbf{F} = \begin{bmatrix} 0 & -z_M & y_M \\ z_M & 0 & -x_M \\ -y_M & x_M & 0 \end{bmatrix} \begin{bmatrix} F\ \cos\alpha_F \\ F\ \cos\beta_F \\ F\ \cos\gamma_F \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & -4 & 2 \\ 4 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix} \begin{bmatrix} 40\ \cos\frac{80\pi}{180} \\ 40\ \cos\frac{60\pi}{180} \\ 40\ 0.848 \end{bmatrix} = \begin{bmatrix} -12.16 \\ -74.00 \\ 46.16 \end{bmatrix} = \begin{bmatrix} M_{\mathrm{O}x} \\ M_{\mathrm{O}y} \\ M_{\mathrm{O}z} \end{bmatrix}$$

Notice: Matlab works with radians and not with degrees.
The magnitude of $\mathbf{M}_\mathrm{O}$ is

$$M_\mathrm{O} = \sqrt{{M_{\mathrm{O}x}}^2 + {M_{\mathrm{O}y}}^2 + {M_{\mathrm{O}z}}^2} = \sqrt{(-12.16)^2 + (-74)^2 + (46.16)^2} = 88.06\,\mathrm{Nm}$$

The direction cosines of $\mathbf{M}_\mathrm{O}$ are

$$\cos\alpha_M = \frac{M_{\mathrm{O}x}}{M_\mathrm{O}} = \frac{-12.16}{88.06} = -0.138 \qquad \ldots\ \alpha_M \doteq\ 98^\circ$$

$$\cos\beta_M = \frac{M_{\mathrm{O}y}}{M_\mathrm{O}} = \frac{-74}{88.06} = -0.84 \qquad \ldots\ \beta_M \doteq\ 147^\circ$$

$$\cos\gamma_M = \frac{M_{\mathrm{O}z}}{M_O} = \frac{46.16}{88.06} = 0.524 \qquad \ldots\ \gamma_M \doteq\ 58^\circ 30'$$

As the components $M_{\mathrm{O}x}, M_{\mathrm{O}y}$, and $M_{\mathrm{O}z}$ of $\mathbf{M}_\mathrm{O}$ are equal to the moments M_x, M_y, and M_z of **F** about x, y, and z axes respectively, the following is valid:

$$M_x = M_{\mathrm{O}x}, \quad M_y = M_{\mathrm{O}y}, \quad M_z = M_{\mathrm{O}z}$$

According to the definition of the moment M_a of the force **F** about an axis a, we have

$$M_\mathrm{a} = \boldsymbol{\lambda}^\mathrm{T}\, \mathbf{M}_\mathrm{O}$$

where the unit vector $\boldsymbol{\lambda}$ of a is

$$\boldsymbol{\lambda} = [\cos\alpha_\mathrm{a}, \cos\beta_\mathrm{a}, \cos\gamma_a]^\mathrm{T}$$

As

$$\cos\gamma_\mathrm{a} = +\sqrt{1 - \cos^2\alpha_\mathrm{a} - \cos^2\beta_\mathrm{a}} = 0.8484$$

we conclude that

$$M_\mathrm{a} = [0.5,\ 0.1736,\ 0.8484] \begin{bmatrix} -12.16 \\ -74.00 \\ 46.16 \end{bmatrix} = 45.92\ \mathrm{Nm}$$

Using MATLAB the solution of the problem is much more convenient. The program for the purpose is called `smoment.m`. It can be found in program package.

Exercise 3.2.2 Moment of a force A force F = 50 N is applied at a point M(8;4;4) [m] and its line of action is described by direction angles $(60°;\ 60°;$ acute angle). Determine the moments M_x, M_y, and M_z of the force **F** about the axes x, y, and z respectively and the moment $\mathbf{M}_\mathrm{a}$ about the axis a passing through the point A(0;2;2) and having direction angles $(30°;\ 0°;$ acute angle).

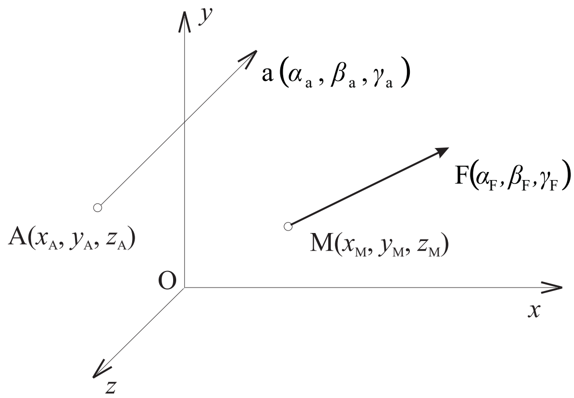

Figure 3.7: Exercise 3.2.2. Moment of the force **F** about the axis a.

Solution

M_x = 41.4 Nm, M_y = -182.8 Nm, M_z = 100 Nm, M_a = -98.47 Nm

Exercise 3.2.3 Moment of a force A force F = 50 N is applied at a point M(2;3;1) [m] and its line of action is described by direction angles $(60°;\ 60°;$ acute angle). The axis a passes through the origin O and lies in xy-plane having the angle $\delta = 30°$ with y axis. Determine

- The moment $\mathbf{M}_{\mathrm{O}}$ of the force **F** about the origin O.
- The moments M_x, M_y, and M_z of the force **F** about the axes x, y, and z respectively.
- The moment M_{a} of the force **F** about the axsis a.

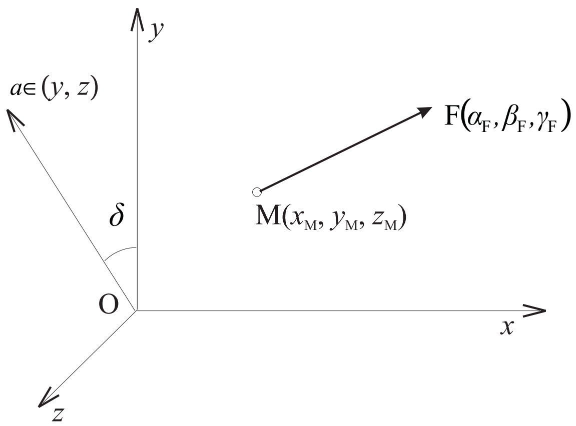

Figure 3.8: Exercise 3.2.3. Moment of the force **F** about the axis a.

Solution
M_x = 81.05 Nm, M_y = -45.7 Nm, M_z = -25 N m, M_{O} = 96.34 Nm,
M_{a} = -52.07 Nm

Exercise 3.2.4 Moment of a force A force F = 40 N is applied at a point M(4;3;-2) [m] and its line of action is described by direction angles $(30°;\ 90°;$ acute angle). Determine the moment $\mathbf{M}_{\mathrm{A}}$ of the force **F** about the point A(-1;3;6) [m] and the moment M_{a} of the force **F** about the axis a passing through the point A and having direction angles $(60°;\ 80°;$ acute angle).

Solution
$M_{\mathrm{A}x}$ = 0 Nm, $M_{\mathrm{A}y}$ = -377.13 Nm, $M_{\mathrm{A}z}$ = 0 Nm, M_{a} = -65.49 Nm

Exercise 3.2.5 Moment of a force A tube is welded to the vertical plate yz at the point A. The tube is loaded by a force F at the point D. The line of action of the force

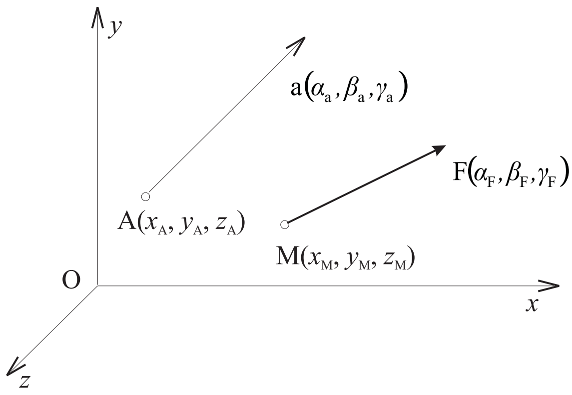

Figure 3.9: Exercise 3.2.4. Moment of the force **F** about the axis a.

F has an angle β with y-axis and an angle γ with z-axis. Determine the moment of the force **F** about the point A. Further determine the moment M_x about x-axis and the moment M_{BC} about BC axis of the same force **F**. How many solutions does the task have? It is known that F = 200 N; $\beta = 60°$; $\gamma = 80°$; $a = 0.2\,\mathrm{m}$; $b = 0.4\,\mathrm{m}$; $c = 0.8\,\mathrm{m}$.

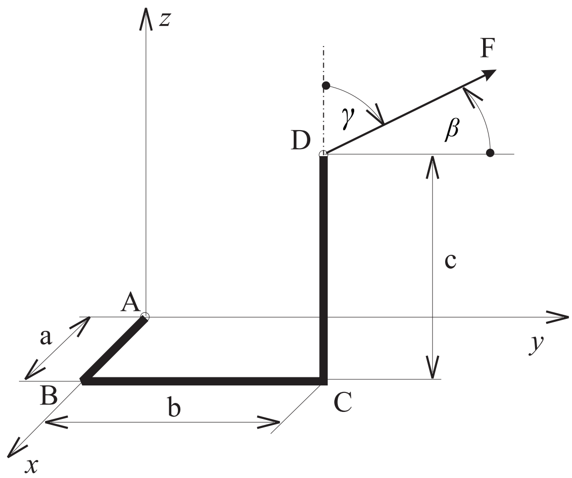

Figure 3.10: Exercise 3.2.5. Moment of the force **F**.

Solution
M_{A} = 152.2 Nm, $M_x = -66.1$ Nm, M_{BC} = 135.75 Nm

Exercise 3.2.6 Moment of a force Determine the moment of the force F = 500 N about the axis AC. The angle $\alpha = 30°$, A(5;0;0), C(0;12;0), E(0;0;20), AB = 6.5, BD = 10 (all dimensions in m).

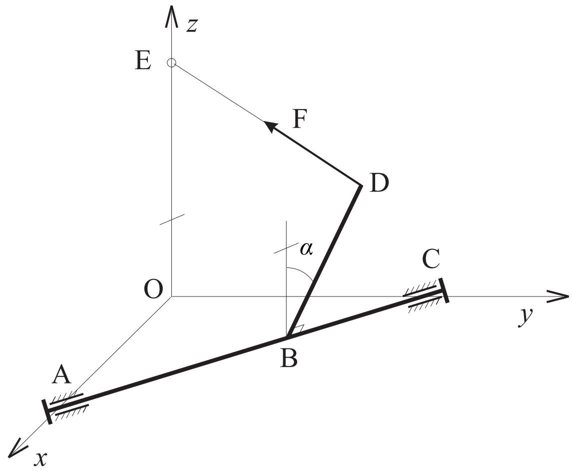

Figure 3.11: Exercise 3.2.6. Moment of a force **F**.

Solution
M_{AC} = 4510 Nm

Exercise 3.2.7 Tangent force A force T = 60 N acts at a tangent of a helix (see Fig.3.12). Find a generic expression for moments M_x, M_y, and M_z of the force **T** about x, y, and z axes respectively as a function of angle φ. Numerically compute the magnitudes of these moments for $\varphi = 750°$ knowing that the radius of the helix is $r = 10$ m and the pitch angle is $\alpha = 30°$. Write a MATLAB program which can be used to calculate the above moments and use it for creating plots of moments values versus angle $0 \leq \varphi \leq 6\pi$.

Solution
M_x = 2222 Nm, M_y = -519.6 Nm, M_z = 3248 Nm

3.2.1 Couples of forces

Two forces **F** and $-$**F** having the same magnitude, parallel lines of action, and opposite sense are said to form a couple of forces, shortly *a couple*.

Moment **C** *of a couple* is a vector perpendicular to the plane of the couple and equal in magnitude to the product of the common magnitude F of the forces and the perpendicular distance d between their lines of action (see Fig. 3.13).

It is called a *couple vector*. Vector **C** is a free vector which may be attached to the origin O (Fig. 3.13) or to any other point.

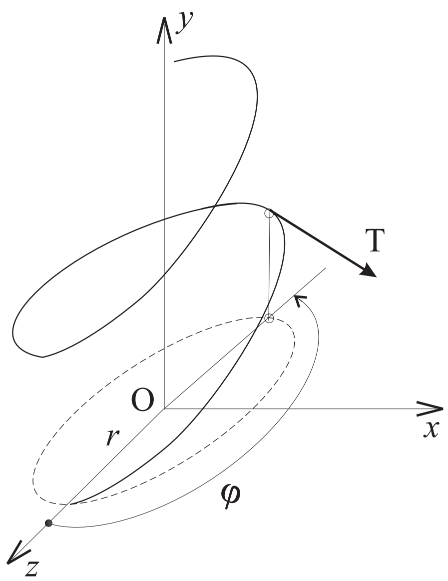

Figure 3.12: Exercise 3.2.7. Tangent force.

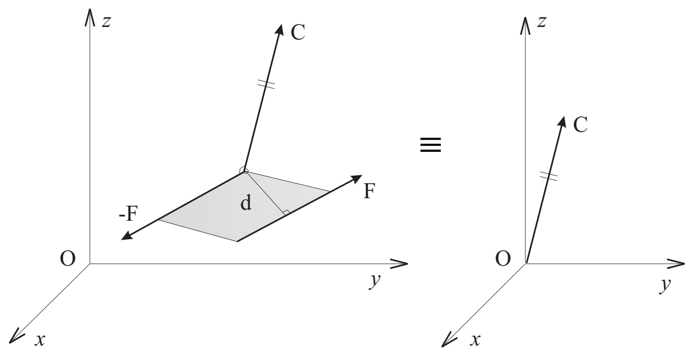

Figure 3.13: Couple of forces.

3.2.2 Principle of transmissibility

The effect of an external force on a rigid body remains unchanged if that force is moved along its line of action.

Warning: This principle is not valid for deformable bodies.

3.2.3 Force systems

Force systems are categorized as follows:

- The most general spatial force system consists of forces the lines of action of which are not parallel to any plane, not all of them are concurrent, and not all of them are parallel.

- The spatial system of parallel forces consists of forces the lines of action of which are all parallel but not all of them lie in one plane.
- The spatial system of concurrent forces consists of forces the lines of action of which intersect at a point but not all of them lie in one plane.
- The most general coplanar force system consists of forces the lines of action of which lie in one plane, not all of them are concurrent, and not all of them are parallel.
- The coplanar system of parallel forces consists of forces the lines of action of which lie in one plane and all of them are parallel.
- The coplanar system of concurrent forces consists of forces the lines of action of which lie in one plane and all of them intersect at a point.
- The collinear force system (the simplest) consists of forces the lines of action of which lie on a common line.

3.2.4 Equivalence of two systems of forces

Any force **F** acting at a point A of a rigid body may be replaced by *a force-couple system* at an arbitrary point O. Force-couple system consists of the force **F** applied at O and a couple moment $\mathbf{C}_\mathrm{O}$ equal to the moment $\mathbf{M}_\mathrm{O}$ about O of the force **F** in its original position (Fig. 3.14).

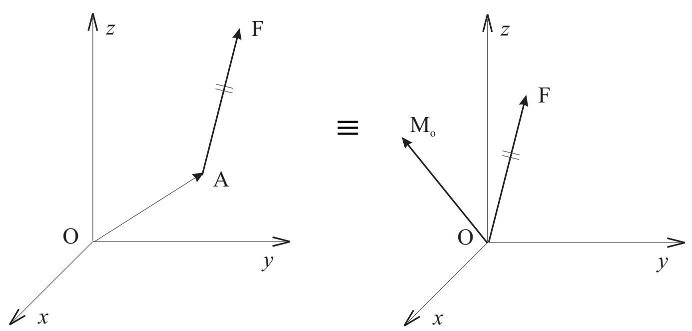

Figure 3.14: Force-couple system.

It follows that any system of forces may be reduced to a force-couple system at a given point O. The resulting force-couple system has the same effect on a given rigid body as the original system of forces (Fig. 3.15).

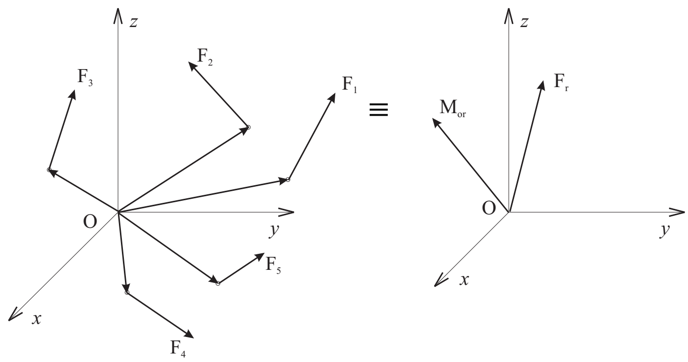

Figure 3.15: Equivalence of two systems of forces.

3.2.5 Equilibrium of force systems

The necessary and sufficient conditions for *equilibrium of a force system* are that the resultant force $\mathbf{F}_{\mathrm{r}}$ and the resultant couple $\mathbf{C}_{\mathrm{r}}$ be zero vectors:

$$\mathbf{F}_{\mathrm{r}} = \mathbf{0}, \qquad \mathbf{C}_{\mathrm{r}} = \mathbf{0} \tag{3.25}$$

The two equations are equivalent to the system of 6 scalar equations:

$$F_{\mathrm{r}x} = 0, \quad F_{\mathrm{r}y} = 0, \quad F_{\mathrm{r}z} = 0 \tag{3.26}$$

$$C_{\mathrm{r}x} = 0, \quad C_{\mathrm{r}y} = 0, \quad C_{\mathrm{r}z} = 0 \tag{3.27}$$

A review of the number and the type of equations ensure equilibrium of a particular force system is shown in Table 3.1.

Remarks:

- An equilibrium equation containing components of forces is called *component force equilibrium equations*.
- An equilibrium equation containing components of moments is called *component moment equilibrium equations*.
- Generally any component force equilibrium equation may be substituted by a component moment equilibrium equation. Care must be taken in choice of an axis or a point for the moment equation. For details consult text books [1], [2], [3].

Force system	**Equations of equilibrium**	**Remarks**
Spatial general	$\sum F_{ix} = 0$ $\sum F_{iy} = 0$ $\sum F_{iz} = 0$ $\sum M_{ix} = 0$ $\sum M_{iy} = 0$ $\sum M_{iz} = 0$	$\sum F_{ix}$ is algebraic sum of x components of the forces; similarly for other axes $\sum M_{ix}$ is algebraic sum of the moments of the forces of the system about x-axis; similarly for other axes
Spatial parallel	$\sum F_{iz} = 0$ $\sum M_{ix} = 0$ $\sum M_{iy} = 0$	forces line of action are paralel to z-axis
Spatial concurrent	$\sum F_{ix} = 0$ $\sum F_{iy} = 0$ $\sum F_{iz} = 0$	
Coplanar	$\sum F_{ix} = 0$ $\sum F_{iy} = 0$ $\sum M_{iz} = 0$	forces lie in xy plane
Coplanar parallel	$\sum F_{iy} = 0$ $\sum M_{iz} = 0$	forces lie in xy plane lines of action are parallel to y-axis
Coplanar concurrent	$\sum F_{ix} = 0$ $\sum F_{iy} = 0$	forces lie in xy plane
Collinear	$\sum F_{ix} = 0$	line of action is x-axis

Table 3.1: Equilibrium conditions.

3.3 Equilibrium of a particle

The particle is a model of a real body. The word "particle" does not imply that the particle is a small body. Modelling a body as particle is equivalent to the assumption that all forces applied on body act at the same point. This assumption is acceptable in many practical engineering applications. The *free particle* and the *constrained particle* should be distinquished. The free particle (such as a planet or a bullet) are rarely encountered in a static equilibrium problems. Most particles are constrained. The first step when solving the equilibrium is "to free" the particle and to sketch so called *free- body diagram*. To free a particle means to isolate it from other bodies which the particle is originally joined or in touch with. All these other bodies must be replaced by forces which they act on the particle in question. After "freeing" the particle we have concurrent system of forces and we solve the problem of equilibrium of this system of forces according to rules described in section 3.2.5. We usually use two component equations of equilibrium in planar (2D) case and three component equations of equlibrium in spatial (3D) case.

Exercise 3.3.1 Sample problem Find a distance d determining the equilibrium position of a collar on a smooth rod (see Fig. 3.16). The free length of a spring is $l_0 = 0.04$ m, the stiffness of a spring is $k = 1000\ \mathrm{Nm}^{-1}$, G = 60 N, $\alpha = 30°$.

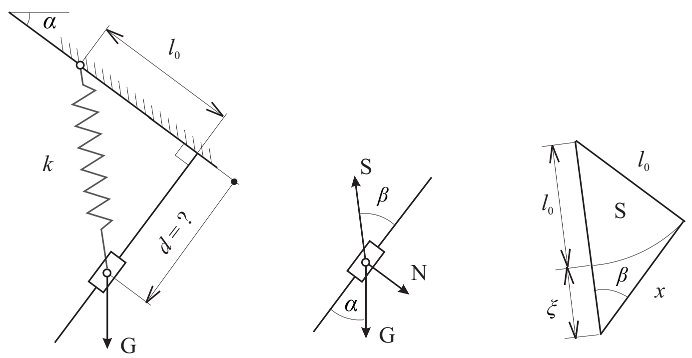

Figure 3.16: Exercise 3.3.1.Equilibrium position of the collar

Solution

A free-body diagram of the collar is on Fig. 3.16 where an auxiliary angle β is introduced. As the collar is supposed to be small, all forces have a common point of action. Therefore we use two independent equilibrium equations for their equilibrium. These are

$$\begin{aligned} G\,\cos\alpha - S\,\cos\beta &= 0 \\ S\,\sin\beta - G\,\sin\alpha - N &= 0 \end{aligned}$$

According to Fig. 3.16 we can express the value of the force **S** in the spring as a function of d:

$$S = k\,\xi = k\,(\sqrt{{l_0}^2 + d^2} - l_0)$$

The function $\cos\beta$ may be expressed as a function of x too:

$$\cos\beta = \frac{d}{\sqrt{{l_0}^2 + d^2}}$$

Substituting the above expressions into the first of equlibrium equations (the second serves to determine N), we have

$$G\,\cos\alpha - k\,(\sqrt{{l_0}^2 + d^2} - l_0)\,\frac{d}{\sqrt{{l_0}^2 + d^2}} = 0$$

and after some manipulations

$$k\,l_0\,d = (k\,d - G\,\cos\alpha)\,\sqrt{{l_0}^2 + d^2}$$

This equation says that for $d > 0$ only such a d has sense which fulfills the inequality

$$k\,d - G\,\cos\alpha > 0$$

this means

$$d > 0.0519\ \mathrm{m}$$

Further manipulation with an equilibrium equation leads to the result

$$k^2\,d^4 - 2\,G\,k\,\cos\alpha\cdot d^3 + G^2\,\cos^2\alpha\cdot d^2 - 2\,G\,k\,{l_0}^2\cos\alpha\cdot d + G^2\,\cos^2\alpha\cdot l_0^2 = 0$$

We have the equation of the fourth degree now. We solve it using Matlab function `roots` as follows:

```
% s211.m
clear all
G=60;k=1000;l0=0.04;alpha=30*pi/180;
calfa=cos(alpha);
c(1)=k*2;c(2)=-2*G*k*calfa;c(3)=G^2*calfa^2;
c(4)=-2*G*k*l0^2*calfa;
c(5)=G^2*calfa^2*l0^2;
d=roots(c)
```

The program supplies four roots from which only $d = 0.0884$ m makes sense. Explain why!

Exercise 3.3.2 Equilibrium of a weight Find a value of a force P for equilibrium of a particle shown on Fig. 3.17 as a function of x. A numerical value of the force compute for $x_{eq} = 0.04$ m. Free lengths of springs are $l_{01} = 0.05$ cm and $l_{02} = 0.08$ m, stifnesses are $k_1 = 5000$ Nm^{-1} and $k_2 = 8000$ Nm^{-1}. Weight $G = 40$ N. No friction is considered.

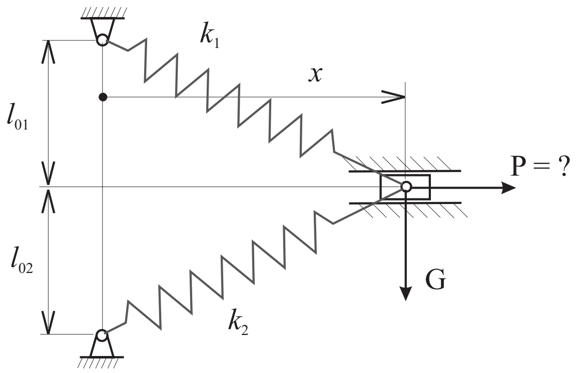

Figure 3.17: Exercise 3.3.2. Equilibrium of the weight

Solution
$P = 77.6$ N

Exercise 3.3.3 Equilibrium of a weight Find x_{eq} such that the equilibrium of the weight W according to Fig. 3.18 is assured. Distance $l = 1$ m, distance $h = 0.4$ m, weights are: $W = 100$ N, $W_1 = 500$ N, $W_2 = 250$ N.
Solution
$x_{\mathrm{eq}} = 0.2$ m

Exercise 3.3.4 Equilibrium position of a particle Find x_{eq} such that the equilibrium position of a particle which follows a smooth curve $y = k\,x^2$ (see Fig. 3.19) is assured. Horizontal force $F = 60$ N, vertical force $G = 100$ N. Dimensions are $a = 0.06$ m, $b = 0.03$ m.
Solution
$x_{\mathrm{eq}} = 0.036$ m

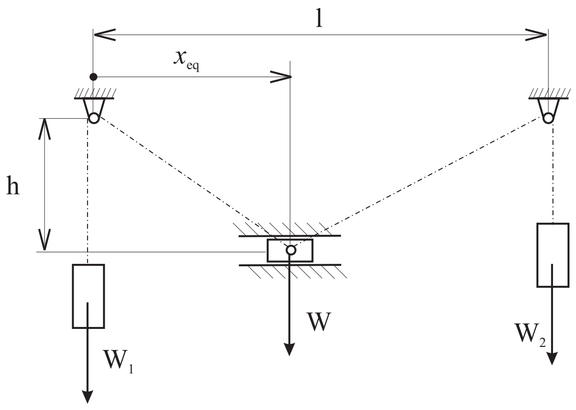

Figure 3.18: Exercise 3.3.3. Equilibrium of the weight

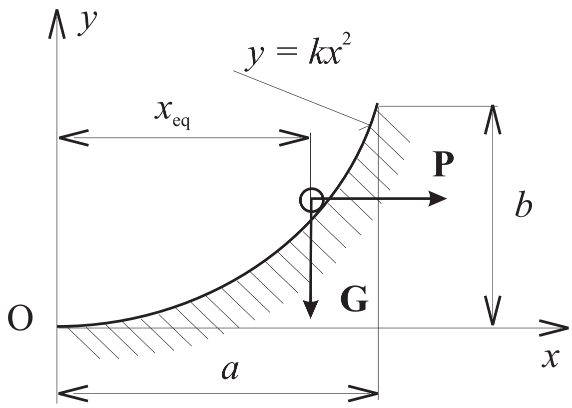

Figure 3.19: Exercise 3.3.4. Equilibrium position of a particle

Exercise 3.3.5 Equilibrium position of a particle Find x_{eq} such that the equilibrium position of a particle located on the end of a massless bar (see Fig. 3.20) is assured. Horizontal force $F = 63$ N, vertical force $G = 90$ N. The length of the bar is $l = 0.08$ m.

Solution

$x_{\text{eq}} = 0.046$ m

Exercise 3.3.6 Equilibrium of a tripod Three bars are connected in such a way that they form a tripod (see Fig. 3.21). A force **F** acts at the common point A_4. Find the forces in the bars. The coordinates of particular points are (in cm): $\mathrm{A}_1(0; 0; 7)$, $\mathrm{A}_2(2.2; 0; 2.5)$, $\mathrm{A}_3(10; 0; 7.5)$, $\mathrm{A}_4(4; 5.5; 6)$, $\mathrm{A}_5(12; 0; 0)$. The magnitude of the force $F = 500$ N.

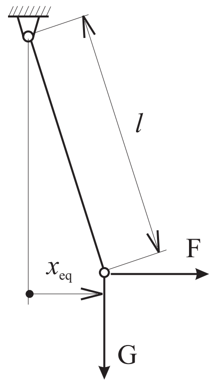

Figure 3.20: Exercise 3.3.5. Equilibrium position of a particle

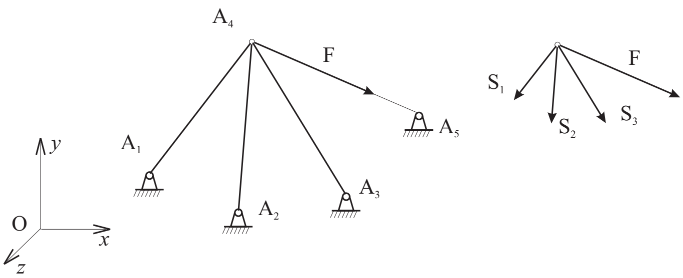

Figure 3.21: Exercise 3.3.6. Equilibrium of a tripod

Solution

We take the common point A_4 as a particle. Three forces from bars and the force **F** act on it. The lines of actions of bar forces lie in the bars' axes.

First of all we draw the free-body diagram supposing tension forces in bars (see Fig. 3.21).

Direction cosines of angles ($\alpha_i,\ \beta_i,\ \gamma_i,\ i = 1,2,3,5$) of lines of action of the particular forces are

$$l_1 = \sqrt{(x_{\mathrm{A1}} - x_{\mathrm{A4}})^2 + (y_{\mathrm{A1}} - y_{\mathrm{A4}})^2 + (z_{\mathrm{A1}} - z_{\mathrm{A4}})^2} = \sqrt{4^2 + 5.5^2 + 1^2} = 6.87\,\mathrm{cm}$$

$$\cos\alpha_1 = \frac{x_{\mathrm{A1}} - x_{\mathrm{A4}}}{l_1} = \frac{-4}{6.87} = -0.5819 \quad ; \quad \cos\beta_1 = \frac{y_{\mathrm{A1}} - y_{\mathrm{A4}}}{l_1} = \frac{-5.5}{6.87} = -0.8002$$

$$\cos\gamma_1 = \frac{z_{\mathrm{A1}} - z_{\mathrm{A4}}}{l_1} = \frac{1}{6.87} = 0.1455$$

$$l_2 = \sqrt{(x_{\mathrm{A2}} - x_{\mathrm{A4}})^2 + (y_{\mathrm{A2}} - y_{\mathrm{A4}})^2 + (z_{\mathrm{A2}} - z_{\mathrm{A4}})^2} = \sqrt{1.5^2 + 5.5^2 + 3.5^2} = 6.69\,\mathrm{cm}$$

$\cos\alpha_2 = \; ^{x}$

Exercise 3.3.7 Equilibrium of a tripod The rope (see Fig. 3.22) goes through frictionless collar connected to three bars A_1A_4, A_2A_4, A_3A_4. The particle at the end of the rope has weight $G = 50$ N. Find the forces in bars. The coordinates of particular points are (in m) $\mathrm{A}_1(0.02; 0; 0)$, $\mathrm{A}_2(0.02; 0.02; 0.07)$, $\mathrm{A}_3(0.08; 0.01; 0.03)$, $\mathrm{A}_4(0.05; 0.07; 0.05)$, $\mathrm{A}_5(0; 0.07; 0)$. Check your result using Matlab program `stripod.m`.

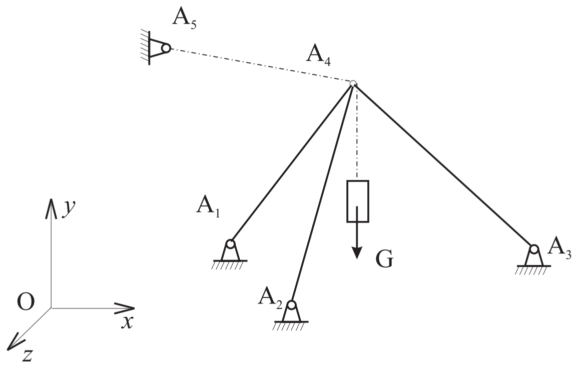

Figure 3.22: Exercise 3.3.7. Equilibrium of a tripod

Solution

$F_1 = -76.7$ N, $F_2 = -6.3$ N, $F_3 = 16.4$ N

Exercise 3.3.8 Equilibrium of a tripod Three bars $\mathrm{A}_1\mathrm{A}_4, \mathrm{A}_2\mathrm{A}_4, \mathrm{A}_3\mathrm{A}_4$ are connected in the point A_4 (see Fig. 3.23). The force $F = 60$ N acts on A_4 parallel to x axis. Find the forces at bars. The coordinates of particular points are (in m) $\mathrm{A}_1(0.02; 0; 0.08)$, $\mathrm{A}_2(0.08; 0; 0.09)$, $\mathrm{A}_3(0.05; 0; 0.03)$, $\mathrm{A}_4(0.04; 0.03; 0.55)$. Check your result using Matlab program `stripod.m`.

Solution

$F_1 = 47.9$ N, $F_2 = -55.5$ N, $F_3 = -7.32$ N

Exercise 3.3.9 Equilibrium of a console System of three bars forming a console support a weight $G = 60$ N located in A_4 (see Fig. 3.24). Find the forces at bars. The coordinates of particular points are (in m): $\mathrm{A}_1(0; 0.04; -0.02)$, $\mathrm{A}_2(0; 0.07; 0)$, $\mathrm{A}_3(0; 0.04; 0.04)$, $\mathrm{A}_4(0.07; 0.04; 0)$. Check your result using Matlab program `stripod.m`.

Solution

$F_1 = -97.1$ N, $F_2 = -152.3$ N, $F_3 = -53.8$ N

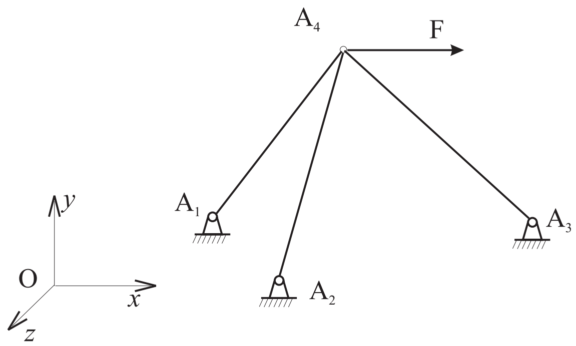

Figure 3.23: Equilibrium of a tripod

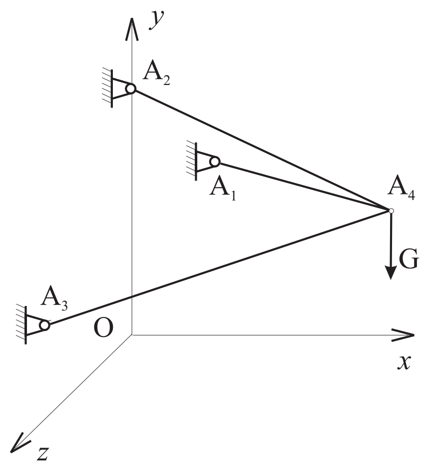

Figure 3.24: Exercise 3.3.9. Equilibrium of a console

Exercise 3.3.10 Equilibrium of a three-bar structure The weight $G = 600\ N$ hangs on three ropes (see Fig. 3.25). Find the forces in the ropes. The coordinates of particular points are (in m): $\mathrm{A}(0.01; -0.01; 0)$, $B(-0.02; -0.02; 0)$, $C(0; 0.02; 0)$, $D(0; 0; -0.02)$. Check your result using Matlab program `stripod.m`.

Solution

$F_\mathrm{A} = 294$ N, $F_B = 207$ N, $F_C = 339$ N

3.4 Equilibrium of a rigid body in a plane

A rigid body is said to be in equilibrium when the sum of external forces (active and reactive too) acting on it forms a system equivalent to zero. For a body this

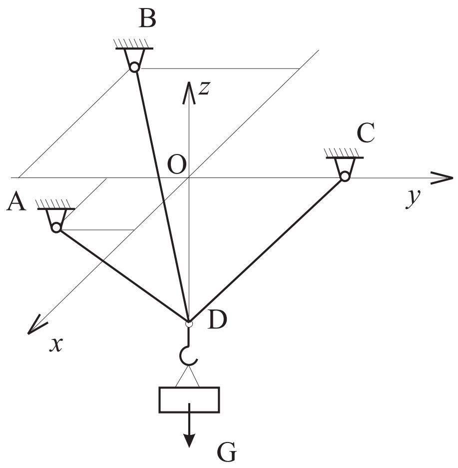

Figure 3.25: Exercise 3.3.9. Equilibrium of a three-bar structure

generally means that

$$\sum_i \mathbf{F}_i = \mathbf{0}, \qquad \sum_i \mathbf{M}_{\mathrm{O}i} = \sum_i (\mathbf{r}_i \times \mathbf{F}_i) = \mathbf{0} \tag{3.28}$$

These two vector equations may be reduced in the planar case (the basic plane being x, y) to the following three scalar equations written in rectangular components of each force and each moment:

$$\sum_i F_{ix} = 0, \quad \sum_i F_{iy} = 0, \quad \sum_i M_{iz} = 0$$

The equations may be used to determine unknown forces applied to the rigid body in plane or unknown reactions exerted by its support.

These equations may be solved for just three unknowns. If they involve more than three unknowns the body is said to be *statically indeterminate*. If they involve fewer than three unknowns, the body is said to be *partially constrained*.

The statement above is not valid absolutely. The solvability of the three equations depends on the properties of the system matrix.

Generally speaking the problem of the equilibrium of a body is always transformed to the problem of the equilibrium of the system of forces that act on the body. To identify all such forces the *free-body diagram* is essential.

Exercise 3.4.1 Sample problem Determine the reactions at points A, B, C, D as functions of the angle $\beta \in \left(-\frac{\pi}{2}; \frac{\pi}{2}\right)$ (see Fig. 3.26). The rectangular shape body is loaded by the forces $F = 600$ N, $Q = 800$ N and by couple $M = 20$ Nm. The length $a = 0.3$ m.

Solution

The free-body diagram is in Fig. 3.26. The equilibrium equations are

$$\begin{array}{lrcl} \sum F_{ix}: & R_C + R_D\,\cos\alpha & = & -F\,\cos\beta \\ \sum F_{iy}: & S - R_D\,\sin\alpha & = & Q + F\,\sin\beta \\ \sum M_{iE}: & -3\,a\,S & = & M - 2\,a\,Q + F\,\cos\beta\,a\, - 4\,a\,F\,\sin\beta \end{array}$$

or in matrix form

$$\begin{bmatrix} 0 & 1 & \cos\alpha \\ 1 & 0 & -\sin\alpha \\ -3a & 0 & 0 \end{bmatrix} \begin{bmatrix} S \\ R_C \\ R_D \end{bmatrix} = \begin{bmatrix} -F\cos\beta \\ Q + F\,\sin\beta \\ M - 2\,a\,Q + F\,a\,\cos\beta - 4\,a\,F\,\sin\beta \end{bmatrix}$$

The numerical solution of the above equation is accomplished using S323.m file.

We have plots of all reactions as a result. For $\beta = 30°$ they are: $S = R_B = 737.9$ N, $R_C = 107.6$ N, $R_D = -724.2$ N

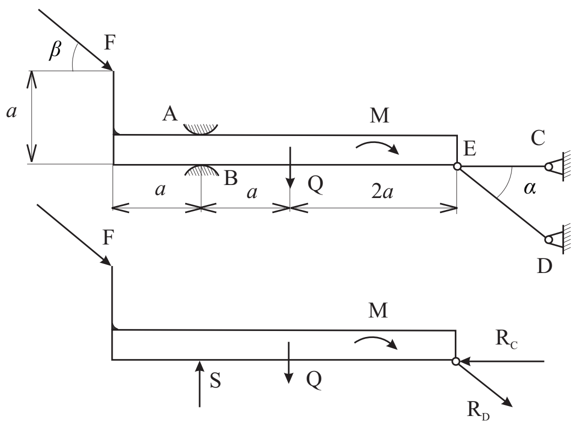

Figure 3.26: Exercise 3.4.1. Equilibrium of a body

Exercise 3.4.2 Equilibrium position of a beam Determine the free length l_0 of a spring the stiffness of which is $k = 50000\mathrm{Nm}^{-1}$. The purpose of the spring is to level the beam loading according to Fig. 3.27. It is known that $G = 600$ N, $M = 12$ Nm, $w_0 = 4000\,\mathrm{Nm}^{-1}$, $a = 0.2$ m. Determine the reaction at A as well.

Solution

$R_A = 397$ N, $l_0 = 0.112$ m

Exercise 3.4.3 Equilibrium position of a plate What is the stiffness k of the spring on Fig. 3.28 to level the plate the weight of which is $G = 150$ N? The free length of the spring is $2\,a = 0.3$ m.

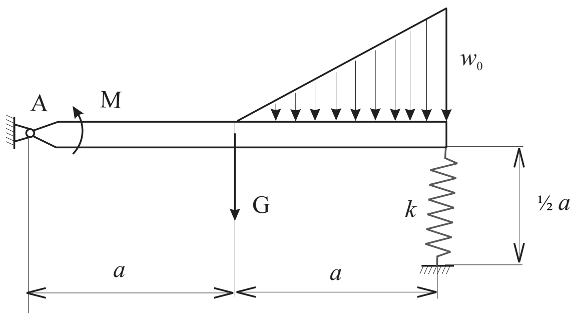

Figure 3.27: Exercise 3.4.2. Equilibrium position of a beam

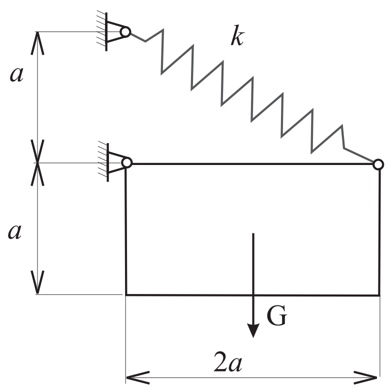

Figure 3.28: Exercise 3.4.3. Equilibrium position of a plate

Solution
$k = 4736\,\mathrm{Nm}^{-1}$

Exercise 3.4.4 Equilibrium of a carriage The weight of the carriage in Fig. 3.29 is $G_1 = 4500$ N, the weight of the load is $G_2 = 2500$ N. The lenght $a = 0.6$ m. The reaction force on the front axle is N_f and the reaction force on the rear axle is N_r. The distance between the front axle and the line of action of G_2 is d. What are the functions $N_\mathrm{f}(d)$ and $N_\mathrm{r}(d)$ for $d \in (a; 2a)$? Find the values of $N_\mathrm{f}(d)$ and $N_\mathrm{d}(d)$ for $d = 1.5\,a$.

Solution
$N_f = 6700$ N, $\;N_r = 300$ N

Exercise 3.4.5 Equilibrium of a crane The crane in Fig. 3.30 is characterized by $G = 10$ kN, $\;a = 1.2$ m, $\;b = 1$ m, $\;c = 8$ m, $\;d = 2$ m. Determine the minimum

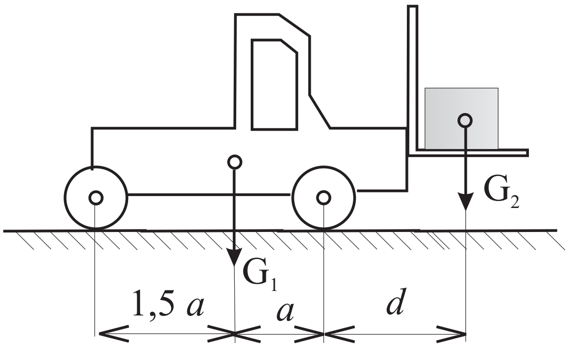

Figure 3.29: Exercise 3.4.4. Equilibrium of a carriage

counterweight $G_{1\min}$ for the crane not to lose its stability with $G_2 = 0$. What is the maximum weight $G_{2\max}$ for the crane not to lose its stability with $G_{1\max}$? Investigate the influence of the position c of the freight G_2 on the magnitudes of the reaction forces. Length c changes in the range from 4 to 8 m.

Hint: The crane loses its stability if either $N_1 \leq 0$ or $N_2 \leq 0$.

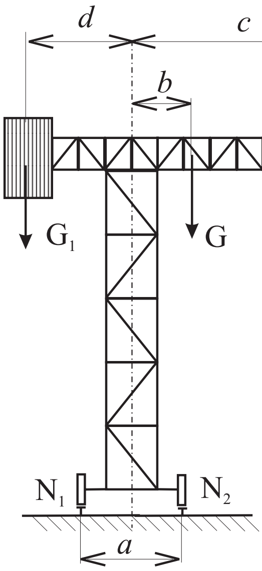

Figure 3.30: Exercise 3.4.5. Equilibrium of a crane

Solution

$G_{1\min} = 1538$ N, $G_{2\max} = 3475$ N

3.5 Equilibrium of a rigid body in space

A rigid body is said to be in equilibrium when the external forces (active and reactive too) acting on it forms a system equivalent to zero.

For a body in space we have

$$\sum_i \mathbf{F}_i = \mathbf{0}, \qquad \sum_i \mathbf{M}_{\mathrm{O}i} = \sum_i \mathbf{r}_i \times \mathbf{F}_i = \mathbf{0} \tag{3.29}$$

These two vector equations are equivalent to the following six scalar equations written in rectangular components of each force and each moment:

$$\sum_i F_{ix} = 0, \quad \sum_i F_{iy} = 0, \quad \sum_i F_{iz} = 0$$

$$\sum_i M_{ix} = 0, \quad \sum_i M_{iy} = 0, \quad \sum_i M_{iz} = 0$$

The equations may be used to determine unknown forces applied to the rigid body in space or unknown reactions exerted by its support.

These equations may be solved for just six unknowns. If they involve more than six unknowns the body is said to be *statically indeterminate*. If they involve fewer than six unknowns, the body is said to be *partially constrained*.

The statement above is not valid absolutely. The solvability of the six equations depends on the properties of the system matrix.

Generally speaking the problem of the equilibrium of a body is always transformed to the problem of the equilibrium of the system of forces that act on the body. To identify all such forces the *free-body diagram* is essential.

From what has been said it follows that the equilibrium of a particular force system is always simpler than the general case. For example the equilibrium of a body in a plane we may solve using three scalar equations only.

Exercise 3.5.1 Sample problem The plate weights 200 N and is supported according to Fig. 3.31. It is loaded by the force **F** the line of action of which has the angle $\delta = 60°$ with the horizontal plane. Determine the of reactions $R_{\mathrm{A}}, R_{\mathrm{B}}$ and the force S in a bar s. It is known that $a = 0.3$ m and $\varepsilon = 45°$.

Solution

The free body diagram is shown in Fig. 3.31. The components of the force **F** are:

$$\begin{aligned} F_x &= -F\cos\delta\cos\varepsilon \\ F_y &= -F\sin\delta \\ F_z &= F\cos\delta\sin\varepsilon \end{aligned}$$

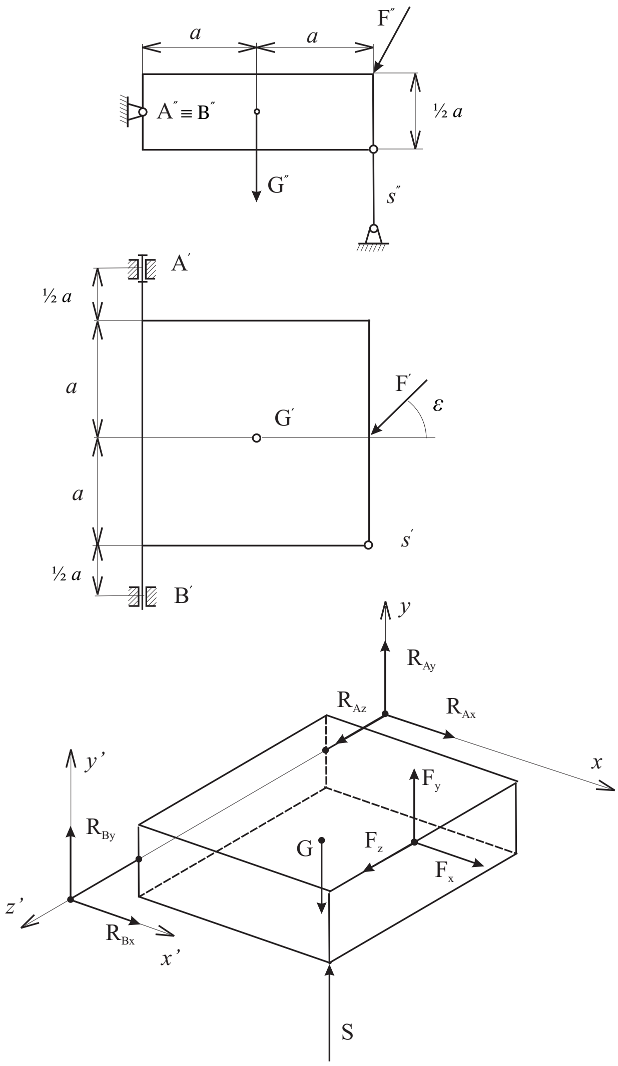

Figure 3.31: Exercise 3.5.1. Equilibrium of a body

The equilibrium of the plate demands the fulfilment of six scalar equations. It is advantageous to use the following set of equations:

$$
\begin{aligned}
\sum F_{iz}: &\quad R_{\mathrm{A}z} + F_z &= 0\\
\sum M_{ix}: &\quad -3\,a\,R_{\mathrm{B}y} + 1.5\,a\,G - 1.5\,a\,F_y + 0.25\,a\,F_z - 2.5\,a\,S &= 0\\
\sum M_{iy}: &\quad 3\,a\,R_{\mathrm{B}x} + 1.5\,a\,F_x - 2\,a\,F_z &= 0\\
\sum M_{iz}: &\quad -a\,G + 2\,a\,S - 0.25\,a\,F_x + 0.2\,a\,F_y &= 0\\
\sum M_{ix'}: &\quad 3\,a\,R_{\mathrm{A}y} - 1.5\,a\,G + 0.5\,a\,S + 1.5\,a\,F_y + 0.25\,a\,F_z &= 0\\
\sum M_{iy'}: &\quad -3\,a\,R_{\mathrm{A}x} - 1.5\,a\,F_x - 2\,a\,F_z &= 0
\end{aligned}
$$

In matrix form we have

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & -2.5 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 3 & 0 & 0 & 0 & 0.5 \\ -3 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} R_{\mathrm{A}x} \\ R_{\mathrm{A}y} \\ R_{\mathrm{A}z} \\ R_{\mathrm{B}x} \\ R_{\mathrm{B}y} \\ S \end{bmatrix} = \begin{bmatrix} -F_z \\ -1.5\,G + 1.5\,F_y - 0.25\,F_z \\ -1.5\,F_x + 2\,F_z \\ G + 0.25\,F_x - 2\,F_y \\ 1.5\,G - 1.5\,F_y - 0.25\,F_z \\ 1.5\,F_x + 2\,F_z \end{bmatrix}$$

The set of six linear algebraic equation we solve using Matlab file s347.m.

Result:

$R_{\mathrm{A}x} = -29.5$ N, $R_{\mathrm{A}y} = 216.6$ N, $R_{\mathrm{A}z} = -176.8$ N, $R_{\mathrm{A}} = 281.1$ N, $R_{\mathrm{B}x} = 206.2$ N, $R_{\mathrm{B}y} = -94.5$ N, $R_{\mathrm{B}} = 266.9$ N, $S = 510.9$ N

Exercise 3.5.2 Equilibrium of a cam shaft The cam shaft in Fig. 3.32 is loaded by the forces $Q = 4\,000$ N, $G = 400$ N, $F = 40$ kN. It is known that $a = 100$ mm, $b = 600$ mm, $T_1/T_2 = 5$. Determine the reaction R_A, R_{B} and the magnitudes of the forces T_1 and T_2.

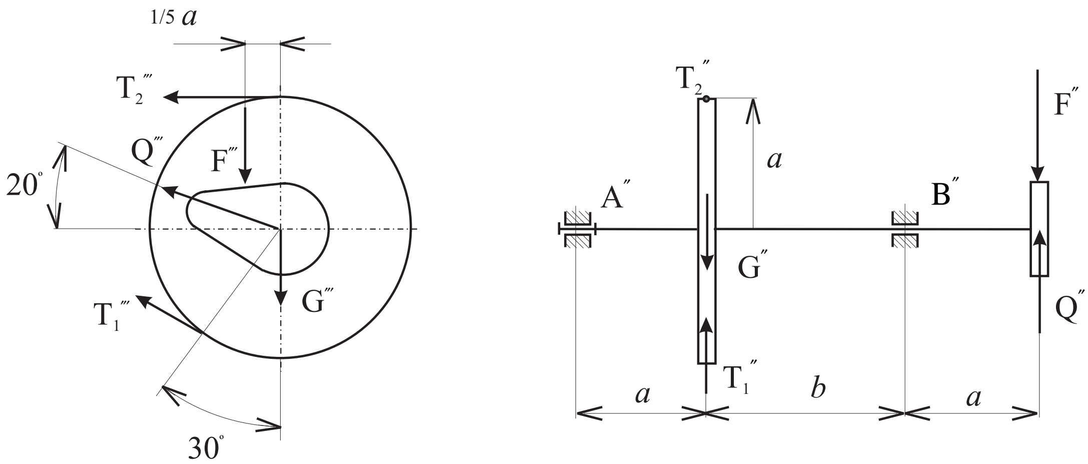

Figure 3.32: Exercise 3.5.2. Equilibrium of a cam shaft

Solution

$R_{\mathrm{A}x} = 0$ N, $R_{\mathrm{A}y} = -946.2$ N, $R_{\mathrm{A}z} = -860$ N, $R_{\mathrm{A}} = 12\,786$ N, $R_{\mathrm{B}y} = 4\,349.4$ N, $R_{\mathrm{B}z} = -581.9$ N, $R_{\mathrm{B}} = 43\,881$ N, $T_1 = 10\,000$ N, $T_2 = 2\,000$ N

Exercise 3.5.3 Equilibrium of a semiaxle One half of a car axle is loaded by the normal component $N = 6$ kN of the reaction, by the tangent component $T_1 = 600$ N of the reaction and by $T_2 = 300$ N transverse component of the reaction. It is known that $a = 100$ mm, $r = 80$ mm, $\alpha = 20°$. Determine the force S and the reaction forces $R_\mathrm{A}, R_\mathrm{B}$ (see Fig. 3.33).

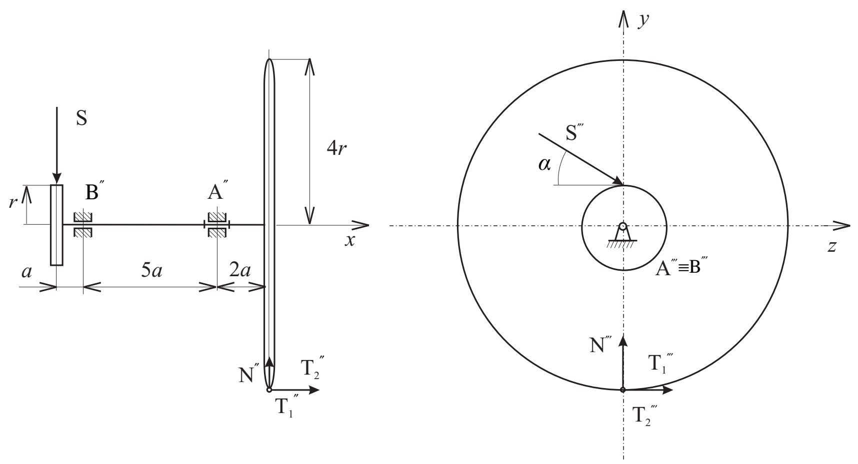

Figure 3.33: Exercise 3.5.3. Equilibrium of a semiaxle

Solution
$R_{\mathrm{A}x} = -300$ N, $R_{\mathrm{A}y} = -8\,767$ N, $R_{\mathrm{A}z} = -360$ N, $R_\mathrm{A} = 8\,779$ N, $R_{\mathrm{B}y} = 3\,640$ N, $R_{\mathrm{B}z} = -2\,640$ N, $R_\mathrm{B} = 4\,497$ N, $S = 2\,554$ N

Exercise 3.5.4 Equilibrium of a stool The stool (see Fig. 3.34) is loaded by the weight $Q = 800$ N of a person sitting eccentrically. The bottom ends of the stool legs are located uniformly on a circle the radius of which is $R = 250$ mm. The eccentricity $r = 100$ mm. Determine the functions $R_\mathrm{A}(\varphi), R_\mathrm{B}(\varphi), R_C(\varphi)$ for $\varphi \in< 0°; 360°)$. Extract the computed values of the reactions for $\varphi = 25°$.

Solution
$R_\mathrm{A} = 510$ N, $R_\mathrm{B} = 298$ N, $R_C = 142$ N

Exercise 3.5.5 Equilibrium of a rotating crane The rotating crane (see Fig. 3.35) is loaded by the force $Q = 4$ kN. The length $a = 1.5$ m. Determine the values of the reactions $R_\mathrm{A}, R_\mathrm{B}, R_C$ as functions of an angle $\alpha \in (-15°; 15°)$. Extract the computed values of the reactions for $\alpha = 9°$.

Solution
$R_\mathrm{A} = 8\,165$ N, $R_\mathrm{B} = 1\,202$ N, $R_C = 3\,488$ N

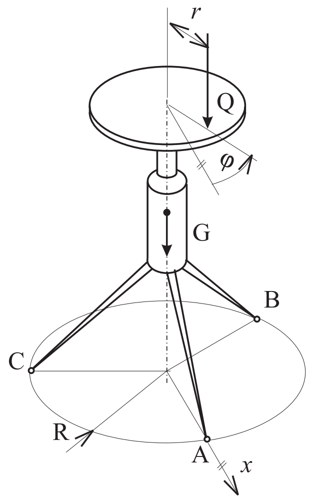

Figure 3.34: Exercise 3.5.4. Equilibrium of a stool

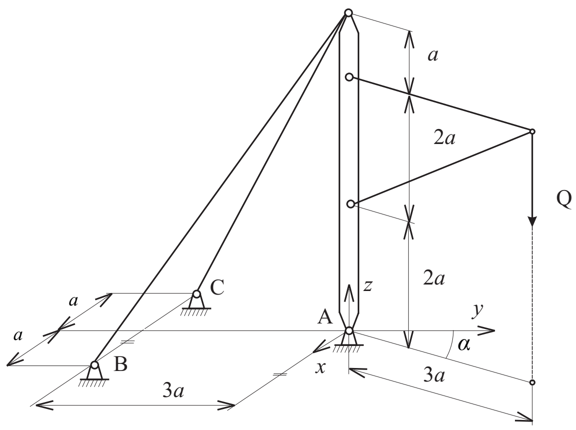

Figure 3.35: Exercise 3.5.5. Equilibrium of a rotating crane

Exercise 3.5.6 Equilibrium of a rod The rod in Fig. 3.36 is loaded by forces $Q = 300$ N, $G = 200$ N. The length $a = 200$ mm. Determine the magnitudes of the forces S_1, S_2 in the bars 1, 2 and the magnitude of the reaction R_{A}.

Solution

$S_1 = 1847$ N, $S_2 = 533$ N, $R_{\mathrm{A}} = 1756$ N

Exercise 3.5.7 Equilibrium of a car axle The car axle (see Fig. 3.37) is loaded by forces $N = 2000$ N, $T_1 = 200$ N, $T_2 = 150$ N. The lengths are $a = 0.1$ m,$r = 0.35$ m. Determine the magnitude of the force S necessary for equilibrium

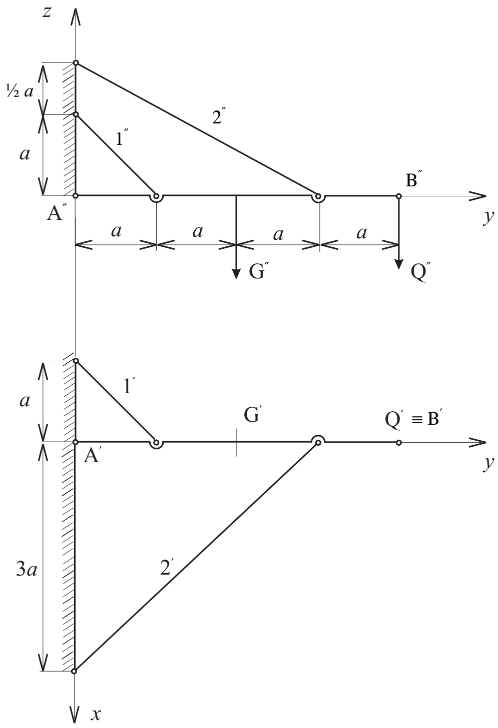

Figure 3.36: Exercise 3.5.6. Equilibrium of a rod

of the axle. Determine also the magnitudes of reaction forces.

Solution

$R_{\mathrm{A}x} = 0, R_{\mathrm{A}y} = 225\ \mathrm{N}, R_{\mathrm{A}} = 1369\ \mathrm{N}, S = 3262\ \mathrm{N}, R_{\mathrm{B}x} = -200\ \mathrm{N}, R_{\mathrm{B}y} = -375\ \mathrm{N}, R_{\mathrm{B}z} = 2612\ \mathrm{N}, R_{\mathrm{B}} = 2647\ \mathrm{N}$

3.6 Systems of rigid bodies

Static analysis of a system of constrained rigid bodies is based on the following theorem: if a system of constrained bodies is in equilibrium each member of the system is in equilibrium as well.

It follows that the equilibrium of the whole system is solved as the equilibrium of each of its member separately. The system consisting of n bodies is substituted by n "freed" bodies. For each body we follow the standard procedure: We draw the free-body diagram and we write down the equilibrium equations.

When solving a mechanism with one degree of freedom, we have in mind that the mobility has to be compensated by an external applied force or couple.

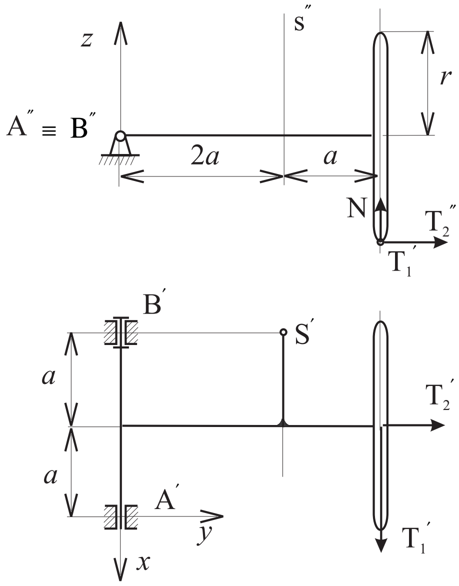

Figure 3.37: Exercise 3.5.7. Equilibrium of a car axle

Exercise 3.6.1 Equilibrium of the shaping machine The mechanism of the shaping machine is loaded by the force $F = 200$ N and by a couple $M_4 = 60$ Nm (see Fig. 3.38). Determine all external reaction forces and the magnitude of the couple M_2 needed for the equilibrium of the mechanism in this particular position $\psi = 45°$ when $l = 0.8$ m, $l_1 = 0.1$ m, $h = 0.28$ m, $h_1 = 0.1$ m, $h_2 = 0.28$ m, $r = 0.1$ m, $l_4 = 0.5$ m, $\alpha = 20°$.

Solution

The mechanism consists of 5 moving members. We sketch the free-body diagram and we write down three scalar equation of equilibrium for each of members 2, 4, 6.

Member 2:

$$\begin{array}{lrcl}
\sum F_{ix}: & R_{\mathrm{A}x} - R_{\mathrm{B}}\,\cos\beta & = & 0\\
\sum F_{iy}: & R_{\mathrm{A}y} - R_{\mathrm{B}}\,\sin\beta & = & 0\\
\sum M_{i\mathrm{A}}: & R_{\mathrm{B}}\,r\,\cos(\psi-\beta) - M_2 & = & 0
\end{array}$$

Member 4:

$$\begin{array}{lrcl}
\sum F_{ix}: & -R_{\mathrm{D}} + N_4\,\cos\beta + R_{\mathrm{C}x} & = & 0\\
\sum F_{iy}: & N_4\,\sin\beta + R_{\mathrm{C}y} & = & 0\\
\sum M_{i\mathrm{C}}: & R_{\mathrm{D}}\,l_4\,\cos\beta - N_4\,p + M_4 & = & 0
\end{array}$$

Member 6:

$$\begin{array}{lrcl}
\sum F_{ix}: & N_6 - F\,\cos\alpha & = & 0\\
\sum F_{iy}: & R_{\mathrm{E}} + R_{\mathrm{H}} - F\,\sin\alpha & = & 0\\
\sum M_{i\mathrm{H}}: & -R_{\mathrm{E}}\,l + F\,(l_1 + l/2)\,\sin\alpha + F\,h_1\,\cos\alpha + N_6\,s & = & 0
\end{array}$$

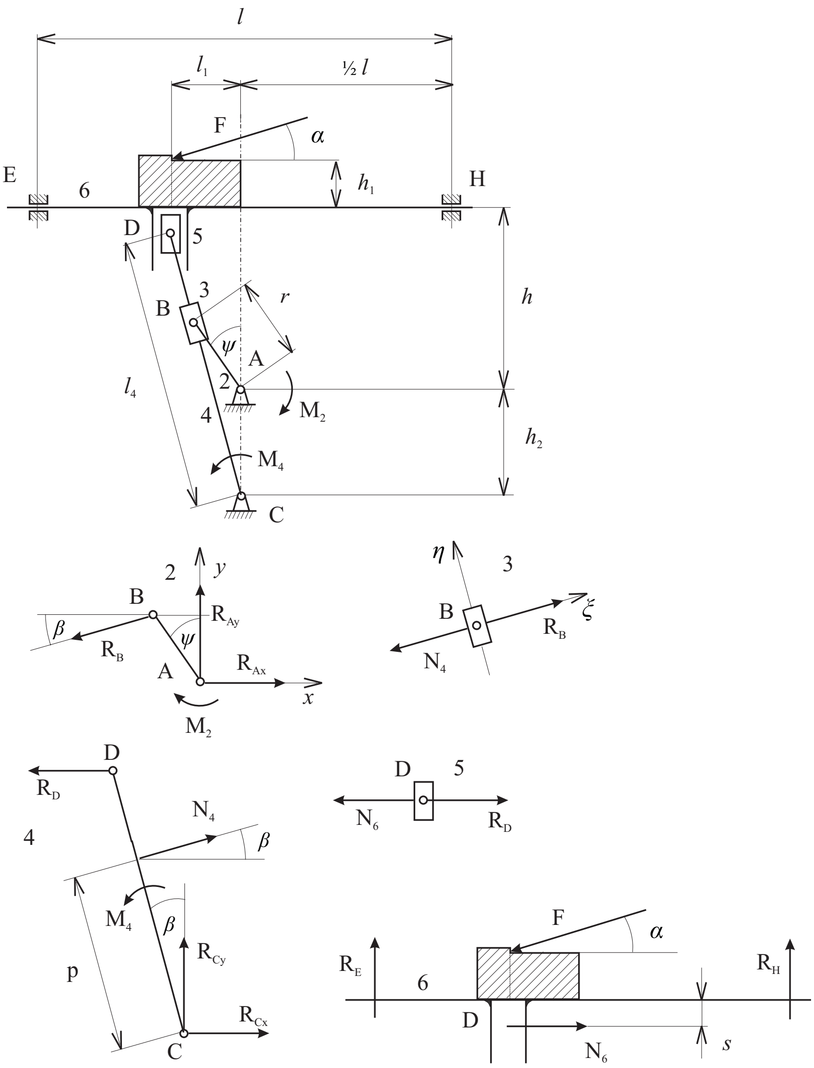

Figure 3.38: Exercise 3.6.1. Equilibrium of the shaping machine

We make use of the special loading pattern of members 3 and 5 - there are just two forces in equilibrium - to write down one equation of equilibrium only:
Member 3:

$$\sum F_{i\xi} : \quad R_{\mathrm{B}} - N_4 \quad = \quad 0$$

Member 5:

$$\sum F_{ix} : \quad R_{\mathrm{D}} - N_6 \quad = \quad 0$$

From geometry we have

$$\beta = \operatorname{arctg} \frac{r \, \sin \psi}{h_2 + r \, \cos \psi}, \qquad p = \frac{h_2 + r \, \cos \psi}{\cos \beta}, \qquad s = h + h_2 - l_4 \, \cos \beta$$

Altogether we have a system of 11 linear algebraic equations in the form

$$\mathbf{A}\,\mathbf{x} = \mathbf{b}$$

System matrix **A** is

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & -\cos\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\sin\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & r\,\cos(\psi-\beta) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\beta & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sin\beta & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -p & l_4\,\cos\beta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & s & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}$$

Vector **x** of unknowns is

$$\begin{bmatrix} M_2 & R_{\mathrm{A}x} & R_{\mathrm{A}y} & R_{\mathrm{B}} & N_4 & R_{\mathrm{D}} & R_{\mathrm{C}x} & R_{\mathrm{C}y} & N_6 & R_{\mathrm{E}} & R_{\mathrm{H}} \end{bmatrix}^{\mathrm{T}}$$

Vector **b** of right-hand side is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -M_4 & F\,\cos\alpha & F\,\sin\alpha & -F\,(l_1+l/2)\,\sin\alpha - F\,h_1\,\cos\alpha & 0 & 0 \end{bmatrix}^{\mathrm{T}}$$

We solve the system of equilibrium equation using Matlab (see program `SSB611S.m`).
Solution
$R_{\mathrm{A}} = 424.7$ N, $R_{\mathrm{C}} = 243.4$ N, $R_{\mathrm{E}} = 82.57$ N, $R_{\mathrm{H}} = -14.17$ N, $M_2 = 35.26$ Nm

Exercise 3.6.2 Equilibrium of a structure A structure according to Fig. 3.39 is loaded by the force $F_4 = 1000$ N and by the couple $M_4 = 100$ Nm. Determine the magnitudes of reactions.

Solution
$R_{\mathrm{A}} = 625$ N, $R_{\mathrm{B}} = 375$ N, $R_{\mathrm{C}} = 1\,151$ N, $R_{\mathrm{D}} = 786.6$ N, $R_{\mathrm{E}} = 258.2$ N

Exercise 3.6.3 Equilibrium of the pliers Determine the force F needed for keeping the cylinder by the force $Q = 100$ N using pliers. The pliers have the parallel moving jaws (see Fig. 3.40). Determine the magnitudes of the reactions as well. It is known that $a = 0.02$ m, $b = 0.04$ m, $c = 0.12$ m, $d = 0.018$ m.

Solution
$F = 25$ N, $R_{\mathrm{A}} = 175$ N, $R_{\mathrm{B}} = 125$ N, $R_{\mathrm{C}} = 25$ N

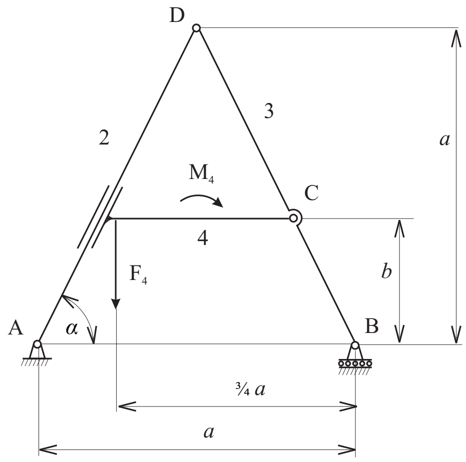

Figure 3.39: Exercise 3.6.2. Equilibrium of a structure

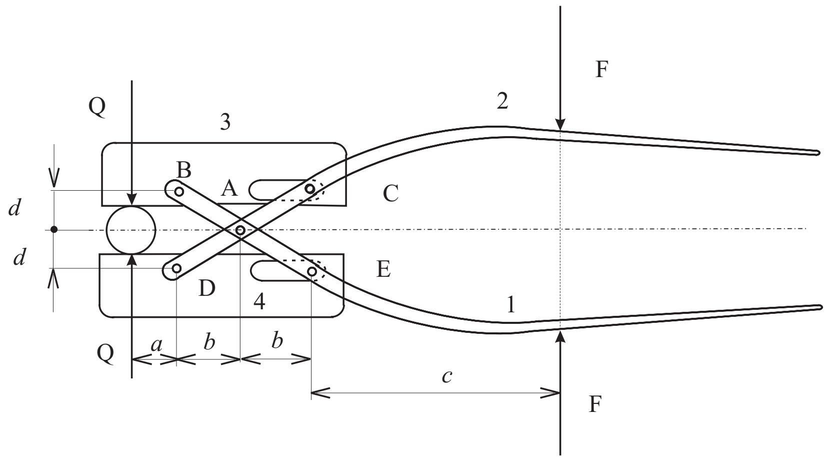

Figure 3.40: Exercise 3.6.3. Equilibrium of the pliers

Exercise 3.6.4 Equilibrium of a landing gear mechanism Determine the pressure p in the hydraulic cylinder of the landing gear mechanism shown in Fig. 3.41. The weight of the wheel is $Q = 500$ N. Determine the reactions $R_\mathrm{A}, R_\mathrm{B}, R_\mathrm{C}$ as well. We know that $d = 0.04$ m, $l_1 = 0.45$ m, $l = 0.8$ m, $h_1 = 0.15$ m, $h = 0.26$ m, $AF = 0.4$ m, $AG = 0.91$ m, $EF = 0.194$ m, $EC = 0.52$ m, $CD = 0.15$ m, $\delta = 120°$.

Solution

$p = 271.7E4\,\mathrm{Nm}^{-2}$, $R_\mathrm{A} = 637.5$ N, $R_\mathrm{B} = 3\,415$ N, $R_\mathrm{C} = 3\,599$ N

Exercise 3.6.5 Equilibrium of decimal scales Determine the force Z needed for

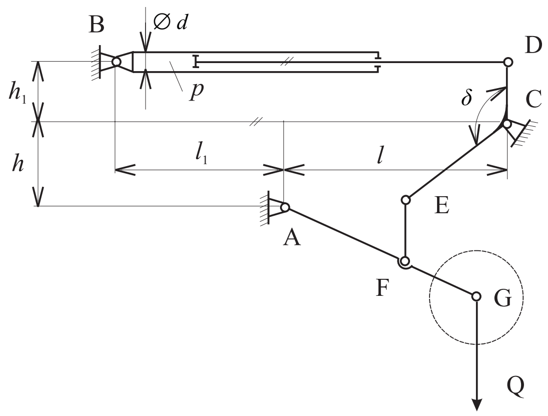

Figure 3.41: Exercise 3.6.4. Equilibrium of the undercarriage

equilibrium of the decimal scales loaded by the force Q (see Fig. 3.42). Determine the reactions $R_{\mathrm{E}}, R_{\mathrm{D}}$ as well. Solve the problem for two different positions of the line of action of the force Q, namely $x_1 = 0.1$ m, $x_2 = 0.3$ m. It is known that $r = 0.2$ m, $s = 0.02$ m, $t = 0.08$ m, $n = 0.1$ m, $v = 0.6$ m, $Q = 800$ N.

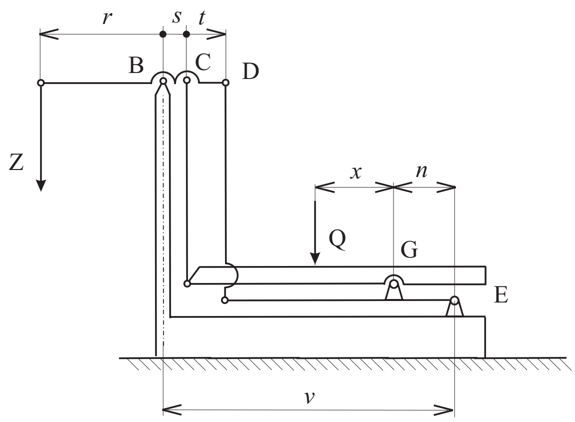

Figure 3.42: Exercise 3.6.5. Equilibrium of a decimal scales

Solution

$Z_1 = 80$ N, $Z_2 = 80$ N, $R_{\mathrm{E1}} = 506.6$ N, $R_{\mathrm{E2}} = 240$ N, $R_{\mathrm{B1}} = 373.3$ N, $R_{\mathrm{B2}} = 640$ N

Exercise 3.6.6 Equilibrium of a lifting platform Lifting of the platform (see Fig. 3.43) is controlled by the force in hydraulic cylinder. Determine the magnitude Z of the force in cylinder in the position shown. Determine the reactions $R_\mathrm{A}, R_\mathrm{B}, R_\mathrm{C}, R_\mathrm{D}$ as well. We know that $l = 1$ m, $a = 0.866$ m, $\varphi = 30°$, $G = 5\,000$ N.

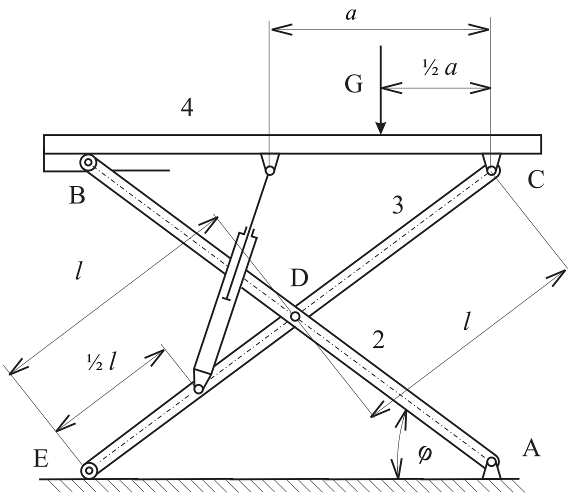

Figure 3.43: Exercise 3.6.6. Equilibrium of a lifting platform

Solution

$Z = 11\,547$ N, $R_\mathrm{A} = 3\,750$ N, $R_\mathrm{B} = 3\,750$ N, $R_\mathrm{C} = 5\,908$ N, $R_\mathrm{D} = 7\,500$ N

Exercise 3.6.7 Equilibrium of a hub lifting mechanism A hub lifting mechanism is loaded by the force $Z_4 = 50$ N (see Fig. 3.44). Compute the magnitude of the force S in spring needed for the equilibrium of the mechanism in the position shown. Determine reactions $R_\mathrm{A}, R_\mathrm{D}$ as well. It is known that $r = 0.15$ m, $\varphi_2 = 30°$.

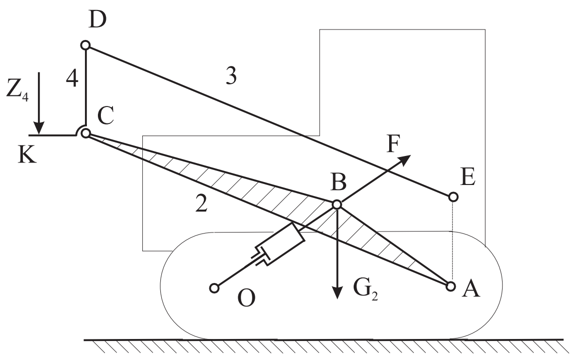

Figure 3.44: Exercise 3.6.7. Equilibrium of a hub lifting mechanism

Solution
$S = 250$ N, $R_{\mathrm{A}} = 250$ N, $R_{\mathrm{D}} = 400$ N

3.7 Trusses

Trusses are an idealized structures consisting of straight and slender rigid bars (members of a truss), each of which is pinned to the rest of the structure. We will limit our attention to the planar trusses, e.g. all bars will lie in one plane. The weights of the members will be neglected. Forces are transmitted from one member to another through smooth pins. The consequence of the idealization described is that members of a truss are so-called "two-force members" which carry only a pair of equal magnitude, oppositely directed forces along their length. We will discuss two methods of solution of trusses:

- method of joints
- method of sections

The method of joints is based on the fact that any pin in the truss has to be in equilibrium. We have two independent equilibrium equation for each pin because the system of forces is a planar concurrent one. We start the solution with the pin on which no more than two unknown forces act. We proceed to another such pin until all unknown forces have been determined. Of course, we can solve all equilibrium equations at the same time as a system of linear algebraic equation.

The method of sections is based on the idea of a division of a truss into two separate parts. Each of them is taken as a body. In case the cut is carried out in such a way that there are three unknown forces. These forces can be found because just three equilibrium equations are available for a general planar force system.

Exercise 3.7.1 Sample problem Using the method of joints, determine the force in each member of the truss shown in Fig. 3.45. State whether each member is in tension or compression. We know that $l = 1$ m, $F_1 = F_2 = 1000$ N, $P = 500$ N.

Solution
A free-body diagram of the entire truss is drawn (see Fig. 3.45). External forces acting on this free body consist of the applied loads F_1, F_2, P and the reactions at A and B. The forces in each member are asumed to be positive (tension). We have

two equilibrium equations for each joint. These equations are

$$
\begin{array}{rcl}
\text{joint A} & : & -R_{\mathrm{A}x} + S_9 \sin\alpha = 0 \\
 & & R_{\mathrm{A}y} + S_1 + S_9 \cos\alpha = 0 \\
\text{joint B} & : & R_{\mathrm{B}x} = 0 \\
 & & R_{\mathrm{B}y} + S_{12} = 0 \\
\text{joint C} & : & -S_9 \sin\alpha - S_8 + S_6 \sin\alpha = 0 \\
 & & -S_{12} - S_9 \cos\alpha + S_7 + S_6 \cos\alpha = 0 \\
\text{joint D} & : & P + S_8 + S_{10} \sin\alpha - S_2 \sin\alpha = 0 \\
 & & S_2 \cos\alpha + S_{11} + S_{10} \cos\alpha - S_1 = 0 \\
\text{joint E} & : & -F_2 - S_2 \sin\alpha = 0 \\
 & & S_2 \cos\alpha + S_3 = 0 \\
\text{joint F} & : & S_4 - S_3 = 0 \\
 & & -S_{11} = 0 \\
\text{joint G} & : & S_5 - S_4 - S_{10} \sin\alpha = 0 \\
 & & -S_7 - S_{10} \cos\alpha = 0 \\
\text{joint H} & : & -S_5 - S_6 \sin\alpha = 0 \\
 & & -F_1 - S_6 \cos\alpha = 0
\end{array}
$$

If we exclude the trival equations

$$R_{\mathrm{B}x} = 0, \quad S_{11} = 0$$

we have the set of 14 linear algebraic equations in matrix form

$$\mathbf{A}\,\mathbf{x} = \mathbf{b}$$

where

$$
\mathbf{A} = \begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & s\alpha & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c\alpha & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & s\alpha & 0 & -1 & -s\alpha & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c\alpha & 1 & 0 & -c\alpha & 0 & -1 \\
0 & 0 & 0 & 0 & -s\alpha & 0 & 0 & 0 & 0 & 0 & 1 & 0 & s\alpha & 0 \\
0 & 0 & 0 & -1 & c\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c\alpha & 0 \\
0 & 0 & 0 & 0 & -s\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c\alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & s\alpha & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -c\alpha & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -s\alpha & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -c\alpha & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \tag{3.31}
$$

$$
\mathbf{x} = \begin{bmatrix} R_{\mathrm{A}x} & R_{\mathrm{A}y} & R_{\mathrm{B}y} & S_1 & S_2 & S_3 & S_4 & S_5 & S_6 & S_7 & S_8 & S_9 & S_{10} & S_{12} \end{bmatrix}^{\mathrm{T}}
$$

$$
\mathbf{b} \;=\; \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -P & 0 & F_2 & 0 & 0 & 0 & 0 & 0 & F_1 \end{bmatrix}^{\mathrm{T}}
$$

We solve the matrix equation using Matlab.

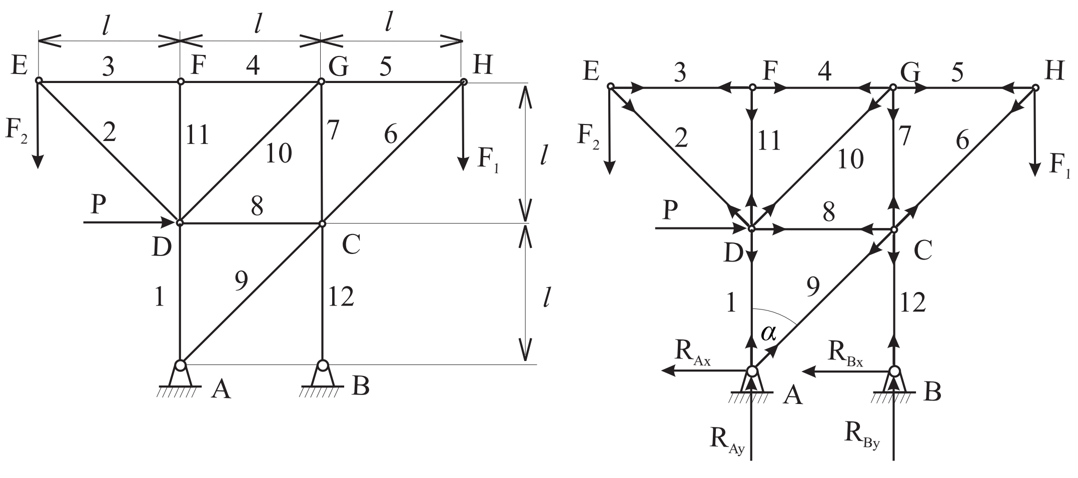

Figure 3.45: Exercise 3.7.1. Equilibrium of a truss

Solution

$R_{\mathrm{A}x} = 500$ N, $R_{\mathrm{A}y} = 500$ N, $R_{\mathrm{B}x} = 0$, $R_{\mathrm{B}y} = 500$ N, $S_1 = -1\,000$ N,

$S_2 = -1\,414$ N, $S_3 = 1\,000$ N, $S_4 = 1000$ N, $S_5 = 1\,000$ N, $S_6 = -1\,414$ N, $S_7 = 0$, $S_8 = -1\,500$ N, $S_9 = 707$ N, $S_{10} = 0$, $S_{11} = 0$, $S_{12} = -500$ N

Exercise 3.7.2 Planar truss Using the method of joints, determine the force in each member of the truss shown in Fig. 3.46. State whether each member is in tension or compression. It is known that $F_1 = 1\,500$ N, $F_2 = 2\,000$ N, $l = 2$ m.

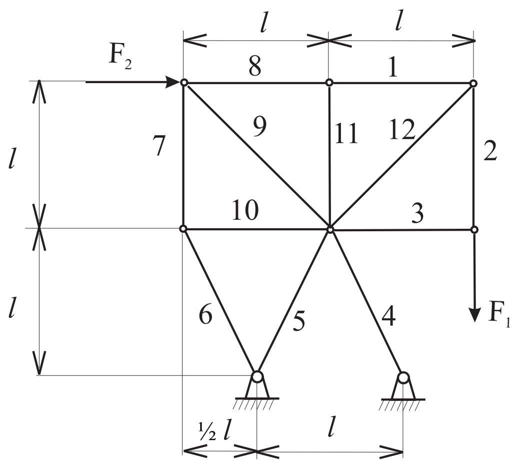

Figure 3.46: Exercise 3.7.2. Equilibrium of a planar truss

Solution
$S_1 = 1\,500$ N,$S_2 = 1\,500$ N, $S_3 = 0$, $S_4 = -6\,988$ N, $S_5 = 1\,398$ N, $S_6 = 3\,913$ N, $S_7 = 3\,500$ N, $S_8 = 1\,500$ N, $S_9 = -4\,950$ N, $S_{10} = -1\,750$ N, $S_{11} = 0$, $S_{12} = -2\,121$ N

Exercise 3.7.3 Equilibrium of a bridge Using the method of joints, determine the force in each member of the truss shown in Fig. 3.47. State whether members are in tension or compression. It is known that $F_1 = 500$ N, $F_2 = 1\,000$ N, $a = 0.6$ m, $\alpha = 30°$.

Solution
$R_A = 803.6$ N, $R_B = 1\,010.4$ N, $S_1 = -1\,630$ N,$S_2 = -1\,630$ N, $S_3 = -1\,630$ N, $S_4 = 1\,060$ N, $S_5 = 0$, $S_6 = 395.3$ N, $S_7 = 790.6$ N, $S_8 = 0$, $S_9 = 530.3$ N

Exercise 3.7.4 Equilibrium of a bridge Using the method of joints, determine the force in each member of the truss shown in Fig. 3.48. State whether members are in tension or compression. It is known that $F_1 = F_2 = F_3 = F_4 = F_5 = 400$ N, $a = 0.4$ m.

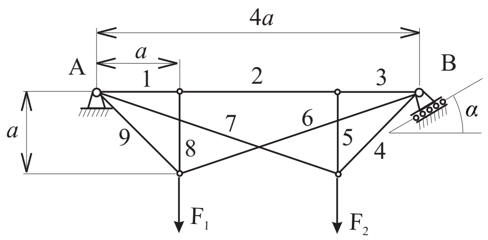

Figure 3.47: Exercise 3.7.3. Equilibrium of a bridge

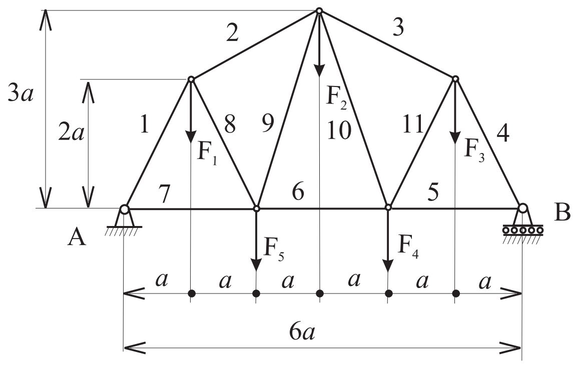

Figure 3.48: Exercise 3.7.4. Equilibrium of a bridge

Solution

$R_{\mathrm{A}} = 1\,000$ N, $R_{\mathrm{B}} = 1\,000.4$ N, $S_1 = -1\,118$ N,$S_2 = -715.5$ N, $S_3 = -715.5$ N, $S_4 = -1\,118$ N, $S_5 = 500$ N, $S_6 = 600$ N, $S_7 = 500$ N, $S_8 = 313.1$ N, $S_9 = 126.5$ N, $S_{10} = 126.5$ N, $S_{11} = 313.1$ N

3.8 Bodies and systems of bodies with friction

The problem of *friction* is the problem of a contact force along contacting surfaces. It is a complex phenomenon and it is difficult to modell it exactly. When solving problems involving friction to distinguish between *sliding contact* and *unmoved contact* is essential.

The simplest modell of friction force gives so called *Coloumb theorem* of friction. This states that (in case of a relative motion of bodies in contact) the following is valid for the friction force $\mathbf{F}_{\mathrm{f}}$ (kinetic friction)

$$\mathbf{F}_{\mathrm{f}} = -\mu_{\mathrm{k}}\, N \frac{\mathbf{v}}{|\mathbf{v}|} \tag{3.32}$$

where $\mathbf{v}$ is vector of the relative velocity of bodies in contact.

The value of a friction force F_{sf} (static friction) when there is no motion between surfaces lies in the interval

$$0 \leq F_{\mathrm{sf}} \leq F_{\mathrm{m}}$$

where

$$F_{\mathrm{m}} = \mu_{\mathrm{s}}\, N$$

Warning: If the motion is not impending, F_{sf} and N should be considered as independent unknowns to be determined from the equilibrium equations. Moreover, the check of the condition

$$F_{\mathrm{sf}} \leq \mu_{\mathrm{s}}\, N$$

must be performed at the end of the solution.

In the above expressions there the following designation is used:

μ_{s} *coefficient of static friction*
μ_{k} *coefficient of kinetic friction*
N normal component of the reaction of the surface

The coefficients of friction depend upon the nature and the condition of the surfaces in contact. They can be found in tables.

3.8.1 Journal bearing

The frictional resistance may be expressed as the magnitude of the couple M which is

$$M_{\mathrm{kj}} = r_{\mathrm{j}}\, \mu_{\mathrm{kj}}\, R \tag{3.33}$$

where r_{j} is the radius of the journal
μ_{kj} is coefficient of kinetic friction in journal
R is reaction of the bearing

3.8.2 Thrust bearing

The frictional resistance of the thrust bearing according to Fig. 3.49 may be expressed as

$$M = \frac{2}{3}\, \mu_{\mathrm{k}}\, \frac{R^3 - r^3}{R^2 - r^2}\, F \tag{3.34}$$

where R is outer radius of the bearing
r is inner radius of the bearing
μ_{k} is coefficient of kinetic friction
F is reaction of the bearing

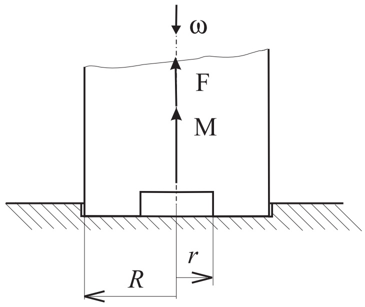

Figure 3.49: Thrust bearing

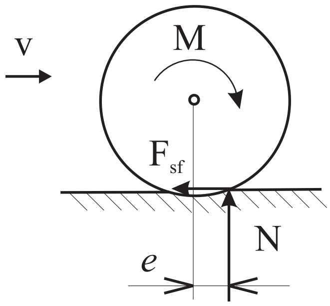

Figure 3.50: Rolling resistance

3.8.3 Rolling resistance

The rolling resistance of the wheel according to Fig. 3.50 may be expressed as

$$M = e\,N \tag{3.35}$$

where e is the coefficient of rolling resistance
N is the normal component of the reaction
F_{sf} is the tangent component of the reaction

3.8.4 Belt friction

For the tensions T_1 and T_2 in two parts of belt or rope slipping around the cylindrical body shape (see Fig.3.51), the following formula is valid

$$\frac{T_2}{T_1} = e^{\mu_{\mathrm{k}}\,\beta} \tag{3.36}$$

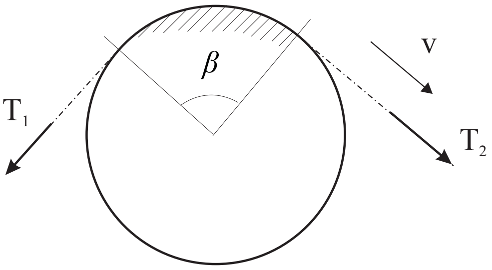

Figure 3.51: Belt friction

where e is the base of natural logarithm
μ_{k} is the coefficient of kinetic friction
β is the angle of contact

The angle of contact must be expressed in radians. The angle β may be larger then 2π. If a rope is wrapped n times around a post, $\beta = 2\pi n$.

Exercise 3.8.1 Sample problem The cam mechanism (see Fig. 3.52) is loaded by forces $Z = 100$ N, $G_2 = 80$ N. Derermine the value of a couple M necessary for equilibrium of mechanism in the position given by $\varphi = 60°$. Further, find the minimum length l_m for mechanism not to get locked. Given values: OS = 0.03 m, r = 0.05 m, h = 0.1 m, l = 0.05 m, e = 0.03 m, n = 0.04 m, r_{j} = 0.01 m, $\mu_{\mathrm{kj}} = \mu$ = 0.1.

Solution

When freeing the particular body we have to keep in mind that the orientation of reaction forces must be estimated. We will use the orintations according to Fig. 3.52.

Member 3:

$$\begin{array}{lrcl}
\sum F_{ix}: & N_1 - N_2 - F_f & & 0 \\
\sum F_{iy}: & -Z - F_{f2} - F_{f1} + N & = & 0 \\
\sum M_{i\mathrm{A}}: & -N\,d + N_2\,(s+l) - Z\,n - N_1\,s & = & 0
\end{array}$$

Member 2:

$$\begin{array}{lrcl}
\sum F_{ix}: & F_f - R_x & = & 0 \\
\sum F_{iy}: & R_y - N - G_2 & = & 0 \\
\sum M_{i\mathrm{O}}: & M - M_{kj} - (G_2 + N)\,\overline{\mathrm{OS}}\,\cos\varphi - F_f\,(r + \overline{\mathrm{OS}}\,\sin\varphi) & = & 0
\end{array}$$

Friction forces are:

$$F_{f1} = |N_1|\,\mu_{\mathrm{k}} \quad , \qquad F_{f2} = |N_2|\,\mu_{\mathrm{k}} \quad , \qquad F_f = |N|\,\mu_{\mathrm{k}} \quad ,$$

$$M_{\mathrm{kj}} = r_{\mathrm{j}}\,\mu_{\mathrm{kj}}\,\sqrt{R_x^2 + R_y^2} \quad .$$

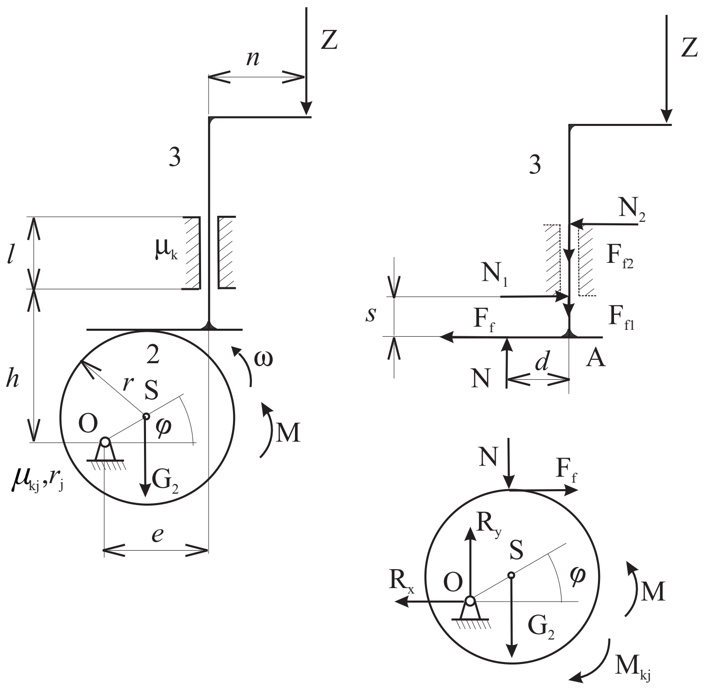

Figure 3.52: Problem 3.8.1. Equilibrium of the cam mechanism

After substitution and some manipulation we have six equations containing 6 unknowns N, N_1, N_2, R_x, R_y, M. First we determine N:

$$N = Z \, \frac{l + 2\,\mu_\mathrm{k}\, n}{(1-{\mu_\mathrm{k}}^2)\, l - 2\,\mu_\mathrm{k}\, d - 2\, s\, {\mu_\mathrm{k}}^2} \quad .$$

and then M:

$$M = r_j\, \mu_\mathrm{kj} \sqrt{(\mu_\mathrm{k}\, N)^2 + (N+G)^2} + (N+G)\,\mathrm{OS}\,\cos\varphi + \mu_\mathrm{k}\, N\,(r + \mathrm{OS}\,\sin\varphi)$$

Geometric relations are:

$$d = \mathrm{OS}\,\cos\varphi \quad , \qquad s = h - r - \mathrm{OS}\,\sin\varphi \quad .$$

The result is M =3.69 Nm

The condition for mechanism not to get *locked* is $N \to \infty$. From the denominator of the expression for N it follows

$$l_m = 2\,\frac{s\,{\mu_\mathrm{k}}^2 + \mu_\mathrm{k}\, d}{1 - {\mu_\mathrm{k}}^2} \quad , \qquad \mu_\mathrm{k} < 1 \quad .$$

The result is l_m= 0.003 m.

Exercise 3.8.2 Equilibrium of a hand-barrow A hand-barrow (see Fig. 3.53) is loaded by forces G_2= 15 N, G_3= 500 N. Determine the magnitude of a force F and the position ψ of its line of action as well assuming that the hand-barrow is moving uniformly in the direction shown. It is known that r= 0.15 m, l= 1.2 m, l_1= 0.4 m, h= 0.15 m, $\beta = 15°$, μ_s= 0.4, μ_{kj}=0.1, r_{j}= 0.01 m, e= 0.005 m.

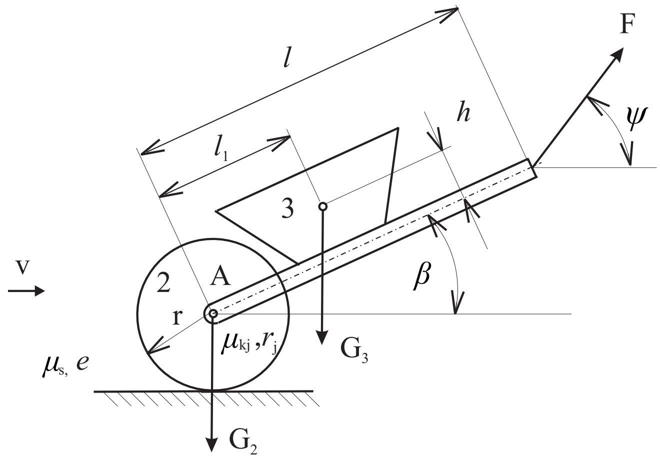

Figure 3.53: Problem 3.8.2. Equilibrium of a hand-barrow

Solution
$F = 154.7N$, $\psi = 84.7°$

Exercise 3.8.3 Equilibrium of weights A rope having weights $Q = 100$ N and $Z = 2Q$ on its ends is thrown over a nonrotating drum and a small pulley at the end of a rotating lever (see Fig. 3.54). Find such a position of the lever for the uniform motion of a rope in the direction shown. Further determine the value of a couple M acting on the lever in this case. Given are $R = 0.1$ m, $r = 0.2$ m, $\mu_{\text{k}} = 0.3$. Neglect the friction between the rope and pulley.

Solution
$\varphi = 192.38°$, $M = 9.54$ Nm

Exercise 3.8.4 Equilibrium of a belt saw A belt-saw (see Fig. 3.55) is prestressed by a force S, so as to prevent against slippage of a belt. Find the minimal value S_{min}

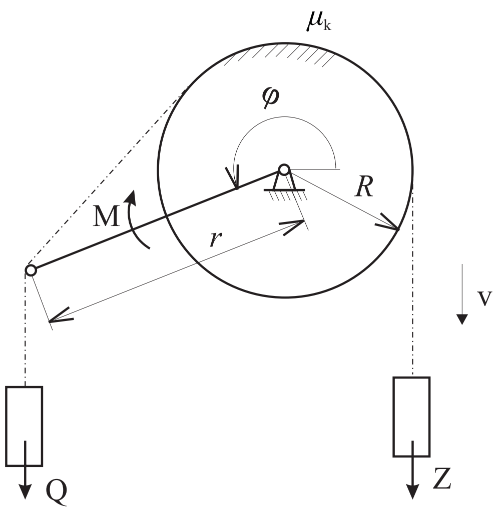

Figure 3.54: Problem 3.8.3. Equilibrium of weights

if a cutting force $O = 100$ N, radii of wheels are $r = 0.2$ m, radii of journals are $r_j = 0.02$ m, $\mu_{kj} = 0.3$, and coefficient of static friction between belt and wheel is $\mu_\mathrm{k} = \mu_\mathrm{s} = 0.2$. Further determine the value of a couple M acting on the bottom wheel.

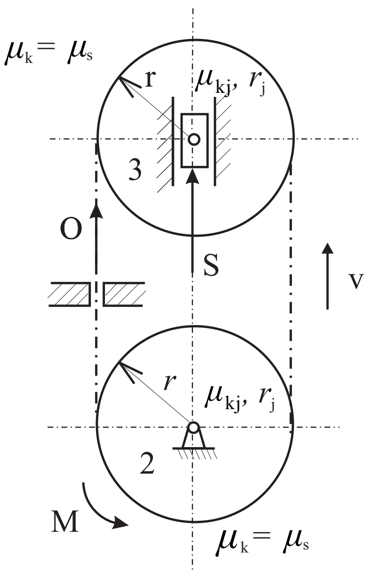

Figure 3.55: Problem 3.8.4. Equilibrium of a belt saw

Solution

$S_{min} = 253.7$ N, $M = 23.6$ Nm

Exercise 3.8.5 Equilibrium of a plate A plate having weight Q is held in its position by a slender rod which can rotate at point A. Determine the maximal value of the angle α (see Fig. 3.56) for the plate of arbitrary Q to be in rest. Given are: $l = 0.3$ m, $\mu_{\mathrm{k}} = \mu_{\mathrm{s}} = 0.15$, $G = 15$ N.

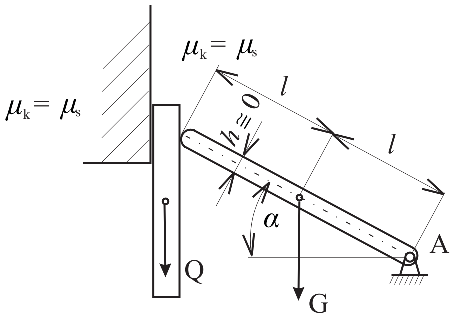

Figure 3.56: Problem 3.8.5. Equilibrium of a plate

Solution
$\alpha = 8.53°$

Exercise 3.8.6 Equilibrium of a brake drum A brake drum weighting $G = 150$ N rotates clokwise and is loaded by a couple $M = 400$ Nm (see Fig. 3.57) . Determine the value of a force P for the uniform rotation of the drum. It is known that $r = 0.25$ m, $l = 0.8$ m, $h = 0.5$ m, $\mu_{\mathrm{k}} = 0.3$, $\mu_{\mathrm{kj}} = 0.05$, $r_{\mathrm{j}} = 0.02$ m.
Solution
$P = 214.7$ N

Exercise 3.8.7 Equilibrium of a roller The roller for tennis court moves uniformly in the direction shown (see Fig. 3.58). Determine the value of a force F and the direction of its line of action. Further find the value of the reaction force between the roller and ground and check the condition for a rolling. Given are $G = 50$ N, $Q = 750$ N, $r = 0.25$ m, $r_1 = 0.02$ m, $l = 0.75$ m, $\mu_{\mathrm{kj}} = 0.1$, $\mu_{\mathrm{s}} = 0.3$, $e = 0.03$ m, $\beta = 30°$.
Solution
$F = 115.8$ N, $\varphi = 40.83°$, $N = 724.3$ N, $F_{sf} = 87.6$ N

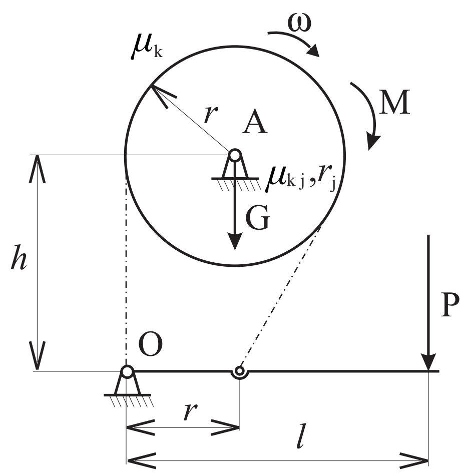

Figure 3.57: Problem 3.8.6. Equilibrium of a brake drum

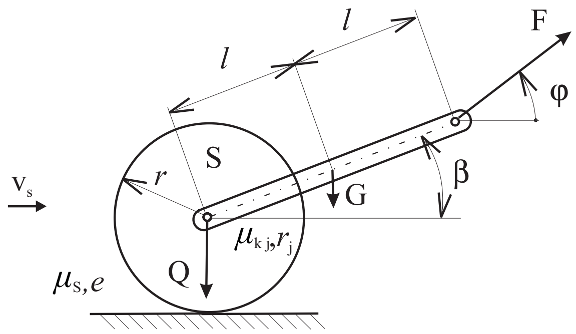

Figure 3.58: Problem 3.8.7. Equilibrium of a roller

3.9 Centre of gravity

The centre of gravity of a rigid body is the point C where a single force **W** called the weight of the body may be applied to represent the effect of the Earth's attraction in any orientation of the body.

In case of a homogeneous body the centre C of gravity coincides with the centroid of the volume V of the body. The coordinates $x_\mathrm{C}, y_\mathrm{C}, z_\mathrm{C}$ of the centroid are defined by the relations

$$x_\mathrm{C}\, V = \int_{(V)} x \,\mathrm{d}V, \quad y_\mathrm{C}\, V = \int_{(V)} y \,\mathrm{d}V, \quad z_\mathrm{C}\, V = \int_{(V)} z \,\mathrm{d}V \tag{3.37}$$

When a body may be divided into i parts having particular centroids $\mathrm{C}_i(x_{\mathrm{C}i}, y_{\mathrm{C}i}, z_{\mathrm{C}i})$

and volumes V_i, the coordinates $x_{\mathrm{C}}, y_{\mathrm{C}}, z_{\mathrm{C}}$ of the whole volume $V = \sum V_i$ are defined by the relations

$$x_{\mathrm{C}}\, V = \sum_i x_i V_i, \quad y_{\mathrm{C}}\, V = \sum_i y_i V_i, \quad z_{\mathrm{C}}\, V = \sum_i z_i V_i \tag{3.38}$$

The same density of particular parts is supposed.

The coordinates of the centroid C of an area or line are defined accordingly. For an area A the following relations are valid:

$$x_{\mathrm{C}}\, A = \int\limits_{(A)} x \,\mathrm{d}A, \quad y_{\mathrm{C}}\, A = \int\limits_{(A)} y \,\mathrm{d}A, \quad z_{\mathrm{C}}\, A = \int\limits_{(A)} z \,\mathrm{d}A \tag{3.39}$$

For a line l the following relations are valid:

$$x_{\mathrm{C}}\, l = \int\limits_{(l)} x \,\mathrm{d}l, \quad y_{\mathrm{C}}\, l = \int\limits_{(l)} y \,\mathrm{d}l, \quad z_{\mathrm{C}}\, l = \int\limits_{(l)} z \,\mathrm{d}l \tag{3.40}$$

The determination of the centroid C is simplified when the line, area or volume possesses certain properties of symmetry.

If the area or line is symmetric with respect to an axis, the centroid C will lie on that axis. If it is symmetric with respect to two axes, C will be located at the intersection of the two axes.

If it is symmetric with respect to a centre O, C will coincide with O.

If a volume possesses a plane of symmetry, its centroid C will lie in that plane. If it possesses two planes of symmetry, C will be located on the line of intersection of the two planes. If it possesses three planes of symmetry intersecting at one point only, C will coincide with that point.

Exercise 3.9.1 Sample problem Locate the centroid of the semicircular area according to Fig. 3.59. Locate the centroid of the boundary line as well.

Solution

Using eq.3.39 we have

$$y_{\mathrm{C}} = \frac{\int\limits_A y\, dA}{A} = \frac{\int\limits_{-r}^{r} dx \int\limits_{0}^{\sqrt{r^2-x^2}} y\, dy}{\frac{\pi r^2}{2}} = \frac{4}{3\pi}\, r = 0.424\, r$$

The centroid of the boundary line is located on y axis due to symmetry. Its y_{C} coordinate we compute from the expresion

$$y_{\mathrm{C}} = \frac{y_{\mathrm{C}1}\, l_1 + y_{\mathrm{C}2}\, l_2}{l_1 + l_2}$$

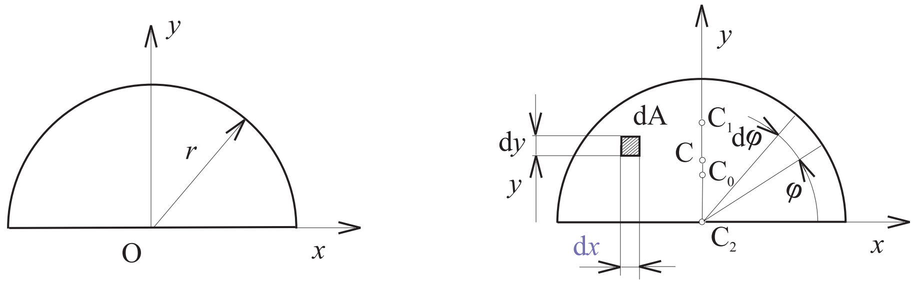

Figure 3.59: Exercise 3.9.1. Center of the area

where $l_1 = 2r$, $l_2 = \pi r$, $y_{C2} = 0$,

$$y_{C1} = \frac{\int\limits_{l1} y\,dl}{l_1} = \frac{\int\limits_0^\pi r\sin\varphi\, r\,d\varphi}{\pi r} = \frac{2}{\pi}\,r = 0.637\,r$$

Therefore

$$y_C = \frac{\frac{2}{\pi}\,r\cdot\pi r + 0\cdot 2r}{\pi r + 2r} = \frac{2r}{\pi+2} = 0.389\,r$$

Exercise 3.9.2 The centroid of a flywheel The centroid of the flywheel should be located in the vertex of the cone (see Fig. 3.60). Determine the height h of the cone.

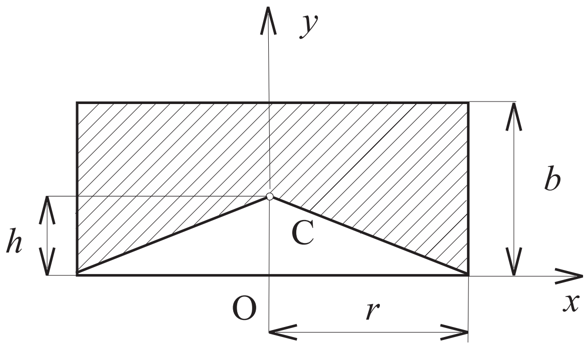

Figure 3.60: Exercise 3.9.2. The centroid of a flywheel

Solution
$h = b\,(2-\sqrt{2}) = 0.586\,b$ m

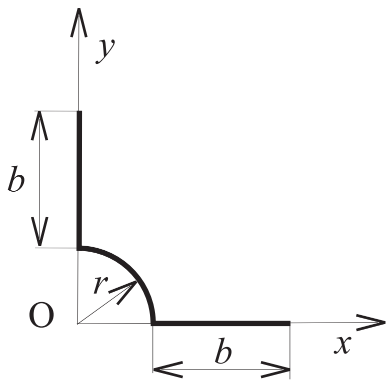

Figure 3.61: Exercise 3.9.3. The centroid of a wire

Exercise 3.9.3 The centroid of a wire Locate the centroid of a wire the shape of which is shown in Fig. 3.61. The lengths are $r = 0.050$ m, $b = 0.04$ m.

Solution

$x_{\mathrm{C}} = y_{\mathrm{C}} = 0.0334$ m

Exercise 3.9.4 The centroid of a stamping Locate the centroid of an area shown in Fig. 3.62 and locate the centroid of the boundary line as well. It is known that $a = 0.07$ m, $b = 0.04$ m, $r = 0.025$ m.

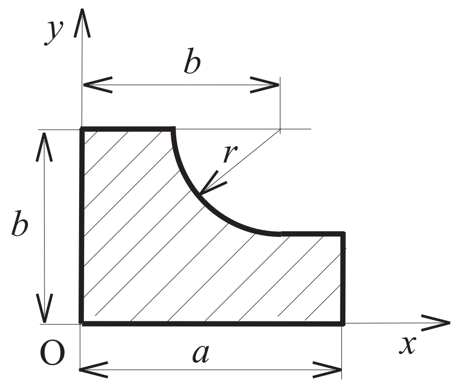

Figure 3.62: Exercise 3.9.4. The centroid of a stamping

Solution

$x_{\mathrm{Carea}} = 0.02714$, $y_{\mathrm{Carea}} = 0.01343$ m, $x_{\mathrm{Ccont}} = 0.02966$, $y_{\mathrm{Ccont}} = 0.01389$ m

Exercise 3.9.5 The centroid of a rivet Locate the centroid of the rivet shown in Fig. 3.63. It is known that $r = 0.03$ m, $h = 0.06$ m, $d = 0.04$ m.

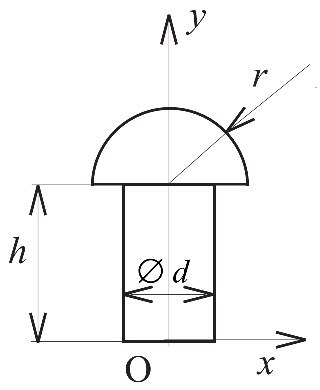

Figure 3.63: Exercise 3.9.5. The centroid of a rivet

Solution
$x_C = 0$, $y_C = 0.0476$ m, $z_C = 0$

Exercise 3.9.6 The centroid of a plate Determine the diameter d_1 of a hole which must be bored for the centroid to be located on y axis (see Fig. 3.64). We know that $d = 0.2$ m, $h = 0.05$ m, $h_1 = 0.04$ m, $x_1 = 0.05$ m.

Solution
$d_1 = 0.0583$ m

3.10 Internal forces in a body

The internal forces in a section of a body are those forces which hold together two parts of a given body separated by the section. Both parts of the body remain in equilibrium. It follows that internal forces which exist at a section are equivalent to all external forces acting on the particular part of the body.

All internal forces in the section are usually replaced by a force-couple system $\mathbf{F}_k, \mathbf{M}_k$ in the centroid C of the cut K. (see Fig.3.65). The force $\mathbf{F}_k$ consists of the *axial force* $\mathbf{N}$ (its line of action is perpendicular to the plane K and *shearing force* $\mathbf{V}$ lying in the plane K. Accordingly, couple M_k consists of two components the first of which is referred to as the *torque* $\mathbf{T}$ (its line of action is perpendicular to the plane K) and the second is called the *bending moment* M_b lying in the plane K.

Now, we will restrict our attention to the case in which a body is loaded in just one plane. Moreover we will analyze the internal forces in a very common

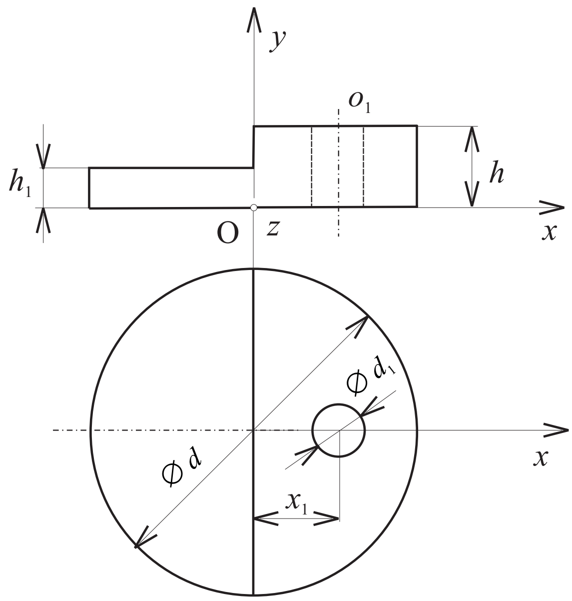

Figure 3.64: Exercise 3.9.6. The centroid of a plate

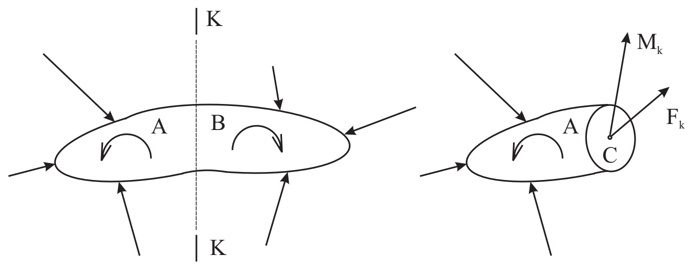

Figure 3.65: Internal forces in a body

engineering structure which is referred to as a *beam*. Beams are usually long straight slender prismatic members designed to support transversal loads. The loads may be either concentrated at specific points, or distributed along the entire length or a portion of the beam. We will limit our analysis to beams which are statically determinate supported. The aim of an analysis is to obtain shear V and bending moment M in all cuts K of the beam.

First we determine the reactions at the supports of the beam. Than we cut the beam at K and use the free-body diagram of one of the two parts of the beam. We adopt the sign convention according to Fig. 3.66. The result of our analysis should be a shear diagram and bending moment diagram representing the shear and the bending moment at any section of the beam. For doing so we use so called

Schwedler theorem saying

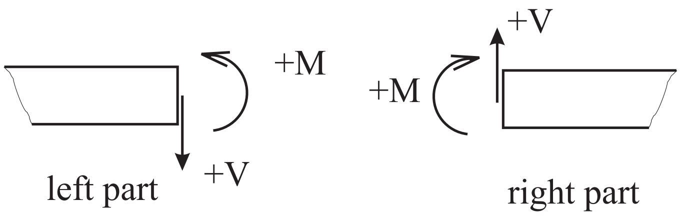

Figure 3.66: Internal forces in a beam

$$\frac{\mathrm{d}V}{\mathrm{d}x} = -w, \qquad \frac{\mathrm{d}M}{\mathrm{d}x} = V \tag{3.41}$$

where w is the distributed load per unit length assumed positive if directed downwards
V is the shear
M is the bending moment
x is the coordinate of the cut oriented from left to right.

We note that the cuts of the beam where the bending moment is maximum or minimum are also the cuts where the shear is zero.

Exercise 3.10.1 Sample problem The beam of the lenght $l = 0.7$ m is shown in Fig. 3.67. It is loaded by the force $F = 400$ N and by partly non-uniformly distributed load characterized by $w_1 = 50\,\mathrm{Nm}^{-1}$ and $w_2 = 400\ \mathrm{Nm}^{-1}$. The angle $\alpha = 45°$. Determine inner forces at section A-A.

Solution

First we determine the force S in the rope. The free-body diagram is shown in Fig. 3.67. The distributed loads are substituted by forces

$$W_1 = l\,w_1 = 35 \text{ N}, \qquad W_2 = l\,\frac{1}{2}\frac{2}{3}\,w_2 = 93.33 \text{ N} \tag{3.42}$$

They act in the centroids of the respective areas.

The moment equilibrium equation with respect to B yields

$$l\,S\sin\alpha - \frac{l}{2}\,W_1 - \frac{2\,l}{9}\,W_2 - \frac{l}{3}\,F = 0 \tag{3.43}$$

and the result is $S = 242.67$ N.

Second we use the free-body diagram of the right-hand part of the beam (see Fig. 3.67) for determining the internal forces N, V, M_o.

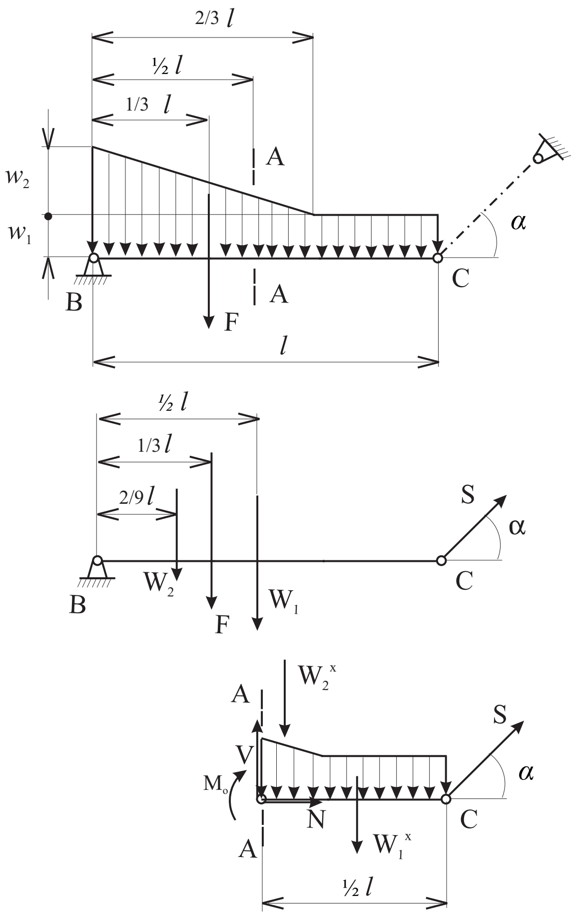

Figure 3.67: Exercise 3.10.1. Internal forces in a beam

We have

$$W_1^x = \frac{1}{2}\,l\,w_1, \qquad W_2^x = \frac{1}{2}\frac{1}{6}\,l\,\frac{1}{4}\,w_2$$

and - for the section A–A -

$$\begin{aligned} N &= S\,\cos\alpha = 171.56\text{ N} \\ V &= S\,\sin\alpha - W_1^x - W_2^x = 148.23\text{ N} \\ M_\mathrm{o} &= \tfrac{l}{2}\,S\,\sin\alpha - \tfrac{l}{4}\,W_1^x - \tfrac{1}{3}\tfrac{1}{6}\,l\,W_2^x = 56.76\ \mathrm{Nm} \end{aligned}$$

Third we construct shear and moment diagrams according to definitions (see Fig. 3.67). The maximum value of the bending moment $M_{\mathrm{omax}} = 34.83\,\mathrm{Nm}$ occurs in the section where $V = 0$, namely where $x = \frac{l}{3} = 0.2$ m from the left side.

Exercise 3.10.2 Internal forces in a beam The simply supported beam (see Fig. 3.68) has the length $l = 0.6$ m, $a = 0.2$ m. It is loaded by the force $F = 200$ N, by the

torque $M = 20\,\mathrm{Nm}$, and by uniformly distributed load $w = 100\,\mathrm{Nm}^{-1}$.
Determine the shear and moment equations for the beam. Draw shear and moment diagrams. Indicate the section where the bending moment reaches its maximum value.

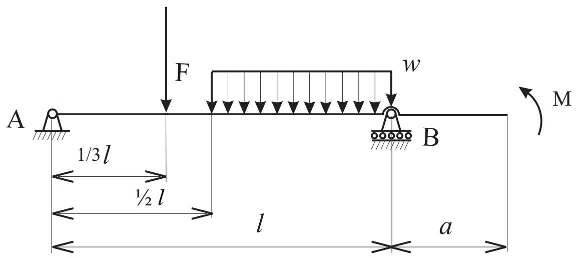

Figure 3.68: Exercise 3.10.2. Internal forces in a beam

Solution
$x = 0.2$ m, $M_{\mathrm{omax}} = 34.83$ Nm.

Exercise 3.10.3 Internal forces in a beam The beam shown in Fig. 3.69 is loaded by the forces $F_1 = 400$ N, $F_2 = 500$ N, by the torque $M = 90\,\mathrm{Nm}$, and by the distributed load $w = 5000\,\mathrm{Nm}^{-1}$. Further we know $a = 0.3$ m, $\alpha = 30°$. Draw shear and moment diagrams. Indicate the section where the bending moment reaches its maximum value and compute it.

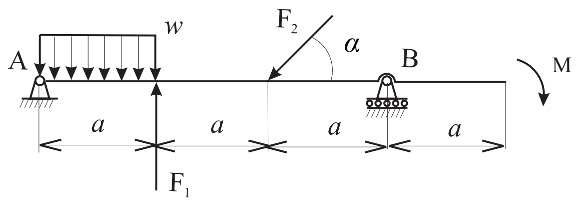

Figure 3.69: Exercise 3.10.3. Internal forces in a beam

Solution
$x = 0.193$ m, $M_{\mathrm{omax}} = 93.44\,\mathrm{Nm}^{-1}$

Exercise 3.10.4 Internal forces in a beam The beam shown in Fig. 3.70 is loaded by the torque $M = 60\,\mathrm{Nm}$, by uniform distributed load $w_1 = 400\,\mathrm{Nm}^{-1}$, and by linearly distributed load $w_2 = 1066\,\mathrm{Nm}^{-1}$. The length $l = 0.3$ m. Draw shear and moment diagrams. Indicate the section where the bending moment reaches its maximum value and compute it.

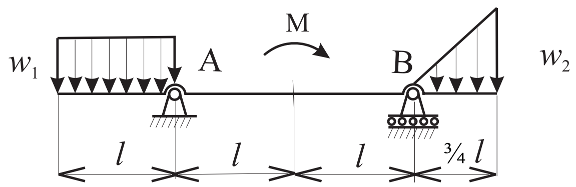

Figure 3.70: Exercise 3.10.4. Internal forces in a beam

Solution
$x = 0.3$ m, $M_{\text{omax}} = -48\,\text{Nm}$

Exercise 3.10.5 Internal forces in a beam The beam shown in Fig. 3.71 is loaded by linearly distributed load $w_0 = 1000\,\text{Nm}^{-1}$. The length $a = 0.1$ m. Draw shear and moment diagrams. Indicate the section where the bending moment reaches its maximum value and compute it.

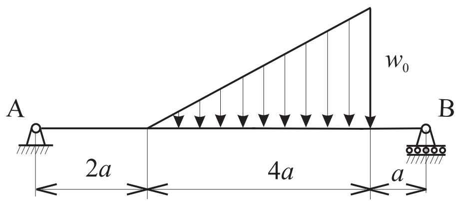

Figure 3.71: Exercise 3.10.5. Internal forces in a beam

Solution
$x = 0.43$ m, $M_{\text{omax}} = 23.59\,\text{Nm}$

Exercise 3.10.6 Internal forces in a beam The simply supported beam according to Fig. 3.72 is loaded by sinus-shape distributed load. It is known that $l = 0.7$ m, $w_0 = 800\,\text{Nm}^{-1}$. Draw shear and moment diagrams. Indicate the section where the bending moment reaches its maximum value and compute it.

Solution
$x = 0.35$ m, $M_{\text{omax}} = 39.72\,\text{Nm}$

3.11 Work and potential energy

Consider a force **F** acting on a particle. The infinitesimal mechanical work dU corresponding to an infinitesimal displacement d**r** of a particle is defined as the

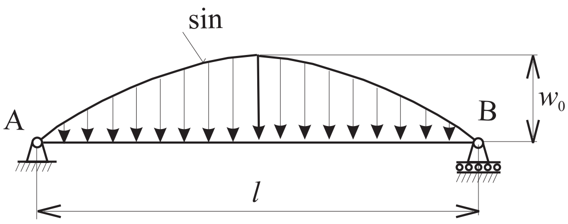

Figure 3.72: Exercise 3.10.6. Internal forces in a beam

scalar product

$$\mathrm{d}U = \mathbf{F} \cdot \mathrm{d}\mathbf{r} \tag{3.44}$$

Denoting respectively by $F = |\mathbf{F}|$ and $\mathrm{d}s = |\mathrm{d}\mathbf{r}|$ the magnitude of the force and the magnitude of displacement, and by α the angle formed by $\mathbf{F}$ and d$\mathbf{r}$, we have

$$\mathrm{d}U = F\,\mathrm{d}s\,\cos\alpha \tag{3.45}$$

The work dU is a scalar quantity and is positive if $\alpha < 90°$, zero if $\alpha = 90°$ and negative if $\alpha > 90°$.

Accordingly we define the infinitesimal work of a couple of moment $\mathbf{M}$ acting on a rigid body as

$$\mathrm{d}U = \mathbf{M} \cdot \mathrm{d}\boldsymbol{\varphi} \tag{3.46}$$

where d$\boldsymbol{\varphi}$ is an infinitesimal angle expressed in radians through which the body rotates.

The work corresponding to a finite displacement of the point of application of the force **F** may be obtained by integration

$$U = \int \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{s_1}^{s_2} F\,\cos\alpha\,\mathrm{d}s \tag{3.47}$$

or (if **M** and d$\boldsymbol{\varphi}$ are parallel vectors)

$$U = \int_{\varphi_1}^{\varphi_2} M\,\mathrm{d}\varphi \tag{3.48}$$

Special attention should be paid to the work of the weight $\mathbf{W}$ of a body of which the center of gravity moves from the height z_1 to z_2 (see Fig. 3.73).

The work for $z_2 > z_1$ is

$$U_{\mathrm{A}_1\to\mathrm{A}_2} = -\int_{z_1}^{z_2} W\,\mathrm{d}z = W(z_1 - z_2) \tag{3.49}$$

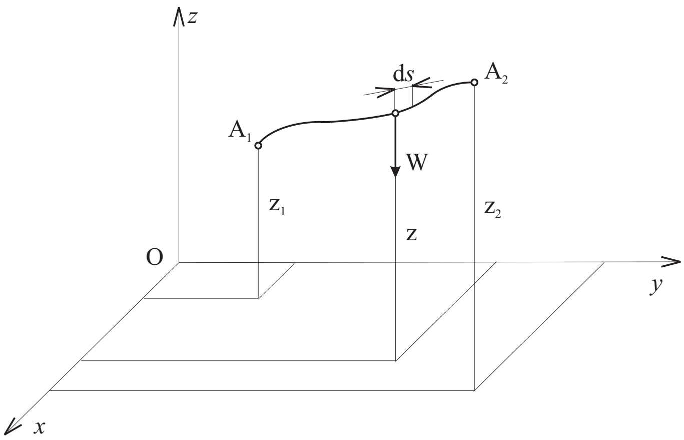

Figure 3.73: Mechanical work of the force S

The work of **W** is positive when the elevation z decreases.

When the work of a force **F** is independent of the path actually followed between A_1 and A_2, the force is said to be a conservative force and its work may be expressed as

$$U_{\mathrm{A}_1 \to \mathrm{A}_2} = V_{\mathrm{A}1} - V_{\mathrm{A}2} \tag{3.50}$$

where V is the potential energy (an ability to exert work) associated with **F**, and $V_{\mathrm{A}1}$ and $V_{\mathrm{A}2}$ represent the values V at A_1 and A_2, respectively.

It follows that the potential energy associated with the force **W** of gravity is generally

$$V = W\,z \tag{3.51}$$

It is clear that force **W** of gravity is conservative.

Another conservative force is that of a spring. The work of the force $F = k\,x$ exerted by a linear spring on a body (see Fig. 3.74) is

$$U_{\mathrm{A}_1 \to \mathrm{A}_2} = -\int\limits_{x_1}^{x_2} k\,x\,\mathrm{d}x = \frac{1}{2}\,k\,(x_1^2 - x_2^2) \tag{3.52}$$

The work is positive when the spring is returning to its undeformed position.

It follows that the potential energy associated with the elastic force **F** is

$$V = \frac{1}{2}\,k\,x^2 \tag{3.53}$$

Exercise 3.11.1 Sample problem A force S acts horizontally on the bar (see Fig. 3.75). The force moves the bar slowly from its initial position to the vertical position. Determine the mechanical work of the force S taking into account friction. We know that $l = 2\,\mathrm{m}$, $r = 1\,\mathrm{m}$, $Q = 5\,\mathrm{N}$, $\mu_k = 0.2$.

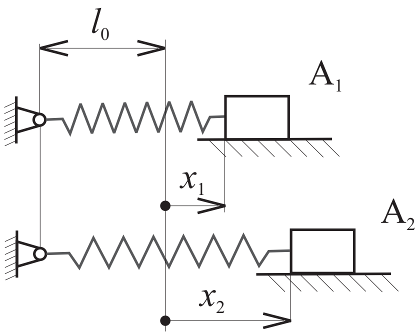

Figure 3.74: Mechanical work of the spring

Solution

First, we sketch the bar in a general position and free it according to Fig. 3.75. The work of the force S is

$$W_S = -\int_{\sqrt{r^2+l^2}}^{r} S\, ds$$

Equilibrium of the bar requires

$$\begin{aligned} S - F_{f1} - F_{f2}\, \cos\varphi - N_2\, \sin\varphi &= 0\ , \\ N_1 - Q + N_2\, \cos\varphi - F_{f2}\, \sin\varphi &= 0\ , \\ N_2\, s\, \cos\varphi - Q\, \frac{l}{2}\, \cos\varphi &= 0 \end{aligned}$$

The friction forces are

$$F_{f1} = N_1\, \mu_{\mathrm{k}}\ , \qquad F_{f2} = N_2\, \mu_{\mathrm{k}}\ ,$$

and

$$\sin\varphi = \frac{r}{s}$$

After substitution and some manipulations we have

$$S = Q\, \mu_{\mathrm{k}} + \frac{Q\, l\, r}{2}\, (1 + {\mu_{\mathrm{k}}}^2)\, \frac{1}{s^2}$$

where s denotes the current position of the bar. The work of the force S is

$$W_S = -\int_{\sqrt{r^2+l^2}}^{r} \left[Q\, \mu_k + \frac{Q\, l\, r}{2}\, (1 + {\mu_{\mathrm{k}}}^2)\, \frac{1}{s^2} \right]\, ds\ ,$$

$$W_S = Q\,\mu_\mathrm{k}\,\left(\sqrt{r^2+l^2}-r\right) + Q\,\frac{l\,r}{2}\,\left(1+{\mu_\mathrm{k}}^2\right)\left(\frac{1}{r}-\frac{1}{\sqrt{r^2+l^2}}\right)$$

Substituting the given numerical values we have the result

$$W_S = 4.11055\ \mathrm{Nm}$$

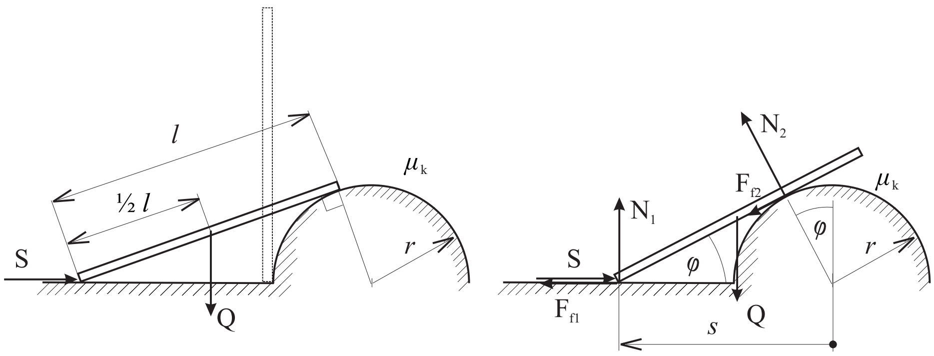

Figure 3.75: Exercise 3.11.1. Mechanical work of the force S

Exercise 3.11.2 Mechanical work of a couple The couple M acts on a drum with a rope wound on it (see Fig. 3.76). The end of the rope is connected to a spring which is stretched. Determine the work W done by the couple M when the spring length is changed about the length $h = 0.1$ m. The weight of the drum is $G = 180$ N, the radius of the drum is $r = 0.08$ m. Take a friction into consideration. The radius of a journal is $r_\mathrm{j} = 0.015$ m, coefficient of friction in the journal is $\mu_\mathrm{kj} = 0.05$, and the spring stiffness is $k = 3000$ N/m.

Solution

$W = 15.3$ Nm

Exercise 3.11.3 Mechanical work of a couple Determine the work of a couple M acting on a screw that presses down a spring about a length $h = 0.03$ m (see Fig. 3.77). The weight of the screw is neglected. The unstressed length of the spring is $l_0 = 0.1$ m, the spring stiffness is $k = 20000$ N/m, the screw-thread is flat, the thread pitch is $\alpha = 10°$, the thread friction is $\mu_\mathrm{k} = 0.05$.

Solution

$W = 11.6$ Nm

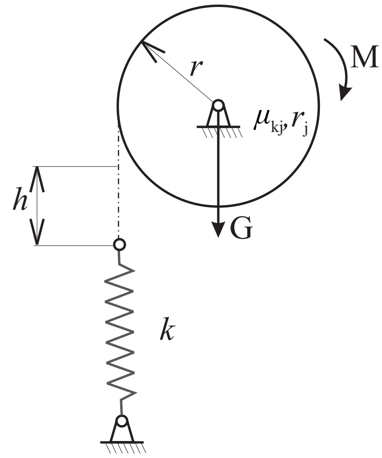

Figure 3.76: Exercise 3.11.2. Mechanical work of the couple M

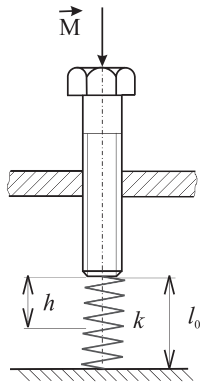

Figure 3.77: Exercise 3.11.3. Mechanical work of a couple M

Exercise 3.11.4 Mechanical work of a force Determine the work of a force P which is necessary to raise the bob from its equilibrium position to the height $h_1 = 0.2$ m (see Fig. 3.78). The line of action of the force P keeps its horizontal position. The weight of the bob is $G = 10$ N, the unstretched length of the spring is $l_0 = 0.3$ m, the stiffness of the spring is $k = 100$ N/m.

Solution

$W = 6$ Nm

Exercise 3.11.5 Mechanical work of a force A cylinder having the weight $G = 300$ N moves under a plate the weight of which is $Q = 500$ N (see Fig. 3.79). Determine the work of a force S acting on the drum centre which is necessary for

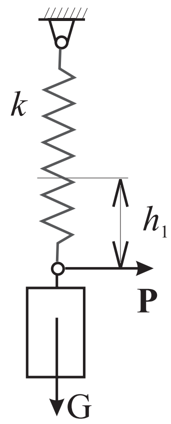

Figure 3.78: Exercise 3.11.4. Mechanical work of a force P

moving the drum through the length h. The coefficient of friction and the coefficient of adhesion between the plate and the drum and between the drum and ground are the same, namely $\mu_{\mathrm{k}} = \mu_{\mathrm{s}} = 0.3$. The length $r = 0.1$ m, e=0.01 m.

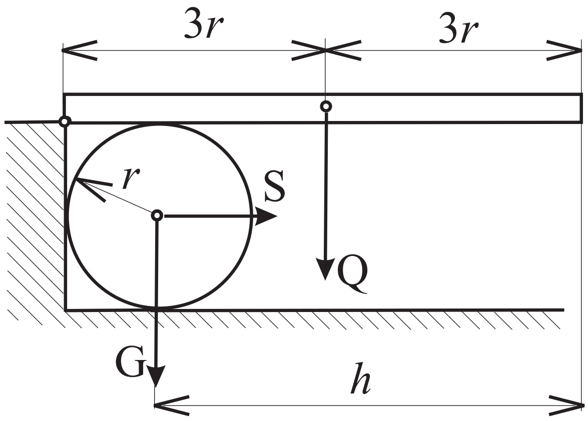

Figure 3.79: Exercise 3.11.5. Mechanical work of a force S

Solution
$W = 203.1$ Nm

Exercise 3.11.6 Mechanical work of a force A cylinder having eccentric centre of mass moves to the right under influence a force P (see Fig. 3.80). Determine the work of a force P which is necessary to move the cylinder about the length $l = 1$ m. The weight of the cylinder is $Q = 80$ N, the radius is $r = 0.3$ m, the coefficient of friction and the coefficient of adhesion between the cylinder and ground is the same, namely $\mu_{\mathrm{k}} = \mu_{\mathrm{s}} = 0.25$. The original position of the cylinder can be seen on Fig. 3.80. The resistance against rolling is neglected.

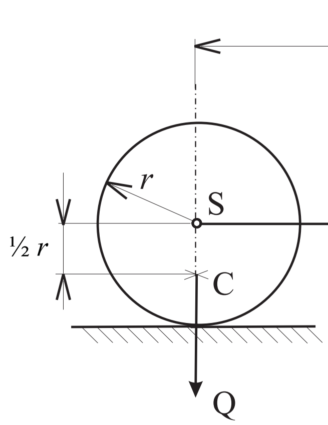

Figure 3.80: Exercise 3.11.6. Mechanical work of a force P

Solution
$W = 18.47$ Nm

3.12 Principle of virtual work

A *virtual displacement* $\delta\mathbf{s}$ of a point is any arbitrary infinitesimal change in the position of the point consistent with the constraints imposed on the motion of the point. This displacement can be just imagined.

Virtual work δU done by a force is defined as $\mathbf{F}^T\delta\mathbf{s}$.

Virtual work δU done by a couple is defined as $\mathbf{M}^T\delta\boldsymbol{\varphi}$.

The *principle of virtual work* (pvw) can be used in statics for solution of equilibrium problem. The following is valid:

The necessary and sufficient condition for the equilibrium of a particle is zero virtual work done by all working forces acting on the body during any virtual displacement $\delta\mathbf{s}$ consistent with the constraints imposed on the particle.

The necessary and sufficient condition for the equilibrium of a rigid body is zero virtual work done by all external forces acting on the particle during any virtual displacement $\delta\mathbf{s}$ consistent with the constraints imposed on the body.

When using the principle of virtual work for a system of connected rigid bodies (mechanism) we must keep in mind that no virtual work is done by internal forces, by reactions in smooth constraints, or by forces normal to the direction of motion. The virtual work is done by reactions when friction is present.

Exercise 3.12.1 Sample problem Using pvw determine the magnitude of a force Z for equilibrium of a crank-slider mechanism in the position given by the angle $\varphi = 30°$. Given is M = 50 Nm, Q = 35 N, r = 0.1 m.

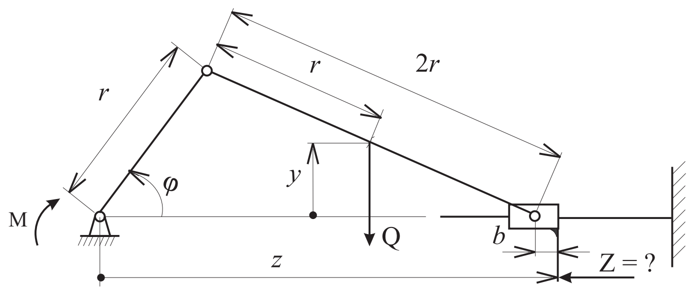

Figure 3.81: Exercise 3.12.1. Equilibrium of the crank-slider mechanism

Solution

First we denote the position of points of action of applied forces Q, Z and the position of the crank by coordinates φ, z, y. According to pvw we can write

$$-M\,\delta\varphi - Q\,\delta y - Z\,\delta z = 0$$

where

$$y = \frac{r}{2}\sin\varphi\,, \qquad \delta y = \frac{r}{2}\cos\varphi\,\delta\varphi$$

$$z = r\,\cos\varphi + r\,\sqrt{4-\sin^2\varphi} + b\,, \qquad \delta z = -r\,\sin\varphi\left(1 + \frac{\cos\varphi}{\sqrt{4-\sin^2\varphi}}\right)\delta\varphi\,\cdot$$

For $\delta\varphi \neq 0$ we have

$$Z = \frac{M + Q\,\frac{r}{2}\cos\varphi}{r\,\sin\varphi\left(1+\frac{\cos\varphi}{\sqrt{4-\sin^2\varphi}}\right)} = \frac{50 + 35\,\cdot\,0,05\,\cdot\,\cos 30°}{0,1\,\cdot\,0,5\left(1+\frac{\cos 30°}{\sqrt{4-\frac{1}{4}}}\right)}$$

The result is

$$Z = 711.92\ \mathrm{N}$$

Exercise 3.12.2 Equilibrium of a mechanism Using pvw determine the magnitude of a couple M acting on the crank when the position of a mechanism (see Fig. 3.82) is given by $\varphi = 30°$. We know that $F = 300$ N, $\alpha = 45°$, $Z = 900$ N, $r = 0.04$ m.

Solution

$M = 42.8$ Nm

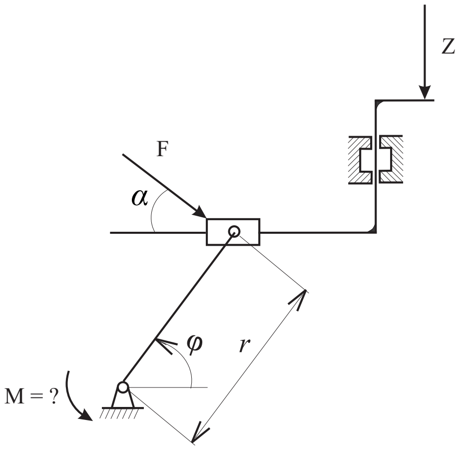

Figure 3.82: Exercise 3.12.2. Equilibrium of a mechanism

Exercise 3.12.3 Equilibrium of a car hood A car hood (see Fig. 3.83) is in equlibrium position given by $\varphi = 30°$. Determine the stiffness k of a spring the free length of which is $l_0 = 0.07$ m. It is known that $Z = 50$ N, $r = 0.1$ m. Use pvw.

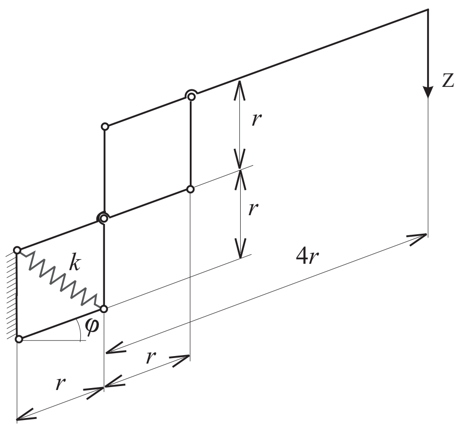

Figure 3.83: Exercise 3.12.3. Equilibrium of a mechnism

Solution
$k = 8333\ \mathrm{Nm}^{-1}$

Exercise 3.12.4 Equilibrium of a mechanism of a front wheel suspension A car wheel suspension (see Fig. 3.84) is loaded by a force $Z = 2500$ N. The spring has a free length $l_0 = 0.1$ m. Using pvw determine the stiffness k of the spring. The length $r = 0.28$ m and the angles are $\varphi = 60°$, $\alpha = 55°$. Determine the stiffness of a spring the free length of which is $l_0 = 0.07$ m. Use pvw.

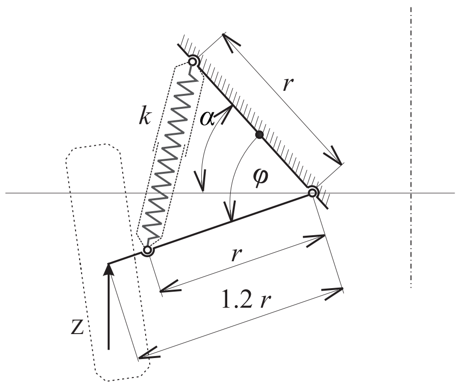

Figure 3.84: Exercise 3.12.4. Equilibrium of a mechanism of a front wheel suspension

Solution
$k = 34509 \text{ Nm}^{-1}$

Exercise 3.12.5 Equilibrium of a bridge The equlibrium position of a draw bridge (see Fig. 3.85) is given by $\varphi = 30°$. Using pvw determine the value of a couple M acting on drum. It is known that $r = 0.1$ m, $l = 4.5$ m, $Q = 5000$ N.

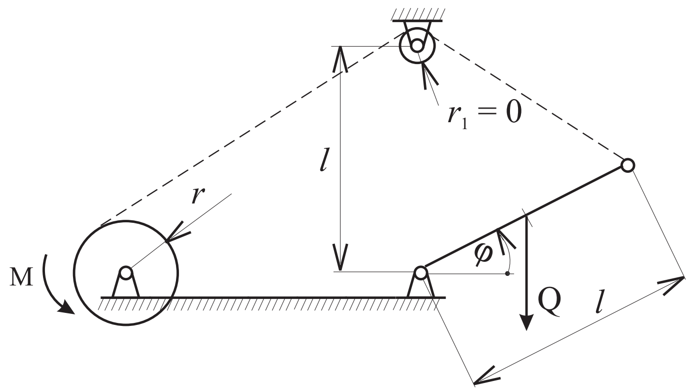

Figure 3.85: Exercise 3.12.5. Equilibrium of a bridge

Solution
$M = 250$ Nm